04

Earlier, we have studied the motion of an object along a straight line. In this chapter, we will study the motion of an object in two dimensions (in plane) and in three dimensions (in space). As a simple case of motion in a plane, we shall discuss motion with constant acceleration and treat it in details as projectile and circular motion.

MOTION IN A PLANE

|TOPIC 1|

Scalars and Vectors

A study of motion will involve the introduction of a variety of quantities that are used to describe the physical world. e.g. Distance, speed, displacement, velocity, acceleration, force, mass, momentum, work, power, energy, etc.

In order to describe the motion of an object to be in two-dimensional and in three-dimensional, we need to understand the concept of vectors first.

Many quantities that have both magnitude and direction need a special mathematical language i.e. the language of vectors to describe such quantities. On the basis of magnitude and direction, all the physical quantities are classified into two groups as scalars and vectors.

CHAPTER CHECKLIST

- Scalar Quantities
- Vector Quantities
- Position and Displacement Vectors
- Multiplication of a Vector by a Real Number
- Resultant Vector
- Addition of Vectors
- Subtraction of Two Vectors
- Resolution of Vectors in plane
- Scalar Product
- Vector Product
- Position, Displacement and Velocity Vectors
- Projectile Motion
- Uniform Circular Motion

SCALAR QUANTITIES

These are the physical quantities which have only magnitude but no direction. It is specified completely by a single number, alongwith the proper unit.

e.g. Temperature, mass, length, time, work, etc.

Note

- The rules for combining scalars follow simple rules of algebra. Scalars can be added, subtracted, multiplied and divided just as the ordinary numbers.
- The quantities with same units can be *added* or subtracted, but the quantities of different units can be multiplied or divided to make sense in scalars.

VECTOR QUANTITIES

These are the physical quantities which have both magnitudes and directions and obey the triangle/ parallelogram laws of addition and subtraction.

It is specified by giving its magnitude by a number and its direction. e.g. Displacement, acceleration, velocity, momentum, force, etc. A vector is represented by a **bold face** type and also by **an arrow** placed over a letter

i.e. $\mathbf{F}, \mathbf{a}, \mathbf{b}$ or $\vec{F}, \vec{a}, \vec{b}$

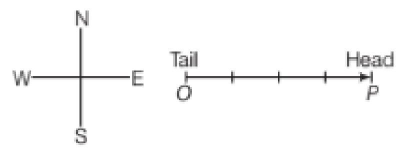
The length of the line gives the magnitude and the arrowhead gives the direction.

Note

The magnitude of a vector is often called its absolute value, indicated by $|\mathbf{v}| = v$.

e.g. Suppose a body has a velocity 40 m/s due East. If 1 cm is chosen to represent a velocity of 10 m/s, a line OP of 4 cm in length and drawn towards East with arrowhead at P will completely represent the velocity of the body.

The point P is called **head** or **terminal point** and point O is called **tail** or **initial point** of the vector **OP**.



Representation of vector OP

Vectors are classified into two types such as

1. Polar Vectors

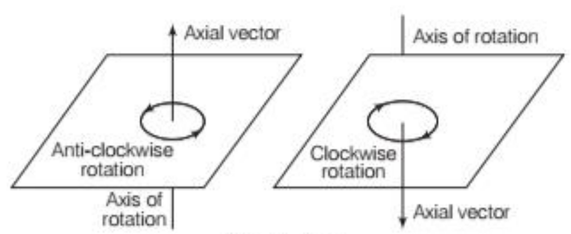
Vectors which have a starting point or a point of application are called **polar vectors.** e.g. Force, displacement, etc.

2. Axial Vectors

Vectors which represent the rotational effect and acts along the axis of rotation are called **axial vectors.**

e.g. Angular velocity, angular momentum, torque, etc.

The axial vector will have its direction along its axis of rotation depending on its anti-clockwise or clockwise rotational effect.



Axial vectors

Note

The physical quantities which have no specified direction and have different values in different directions are called tensors. e.g. Moment of inertia, stress, surface tension, pressure, etc.

Important Definitions Related to Vectors

(i) Modulus of a Vector

The magnitude of a vector is called **modulus** of that vector. For a vector **A**, it is represented by $|\mathbf{A}|$ or A.

(ii) Unit Vector

A vector having magnitude equal to unity but having a specific direction is called a **unit vector.** A unit vector of A is written as $\hat{A}$ and read as A cap. It is expressed as

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A} = \frac{\text{Vector}}{\text{Magnitude of the vector}}$$

$$\boxed{\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}}$$

Hence, any vector can be expressed as the magnitude times the unit vector along its own direction.

In cartesian coordinates, $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along x-axis, y-axis and z-axis.

The magnitude of a unit vector is unity and has no unit or dimensions.

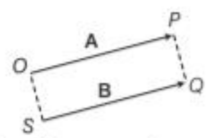
(iii) Null Vector

A vector with magnitude zero and having an arbitrary direction is called a **null vector.**

This vector is also known as **zero vector** and denoted by 0 (zero). e.g. The velocity vector of a stationary object, the acceleration vector of an object moving with uniform velocity.

(iv) Equal Vectors

Two vectors are said to be equal if they have equal magnitude and same direction.



A and **B** are equal vectors

Consider two vectors **A** and **B** which are represented by two equal parallel lines drawn in the same magnitude and direction.

Thus, $\mathbf{OP} = \mathbf{SQ}$ or $\mathbf{A} = \mathbf{B}$

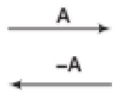
Vectors do not have fixed locations. When we displace a vector parallel to itself, then the vector does not change, such vectors are known as **free vectors**.

e.g. The velocity vector of a particle moving along a straight line is a free vector.

(v) Negative Vector

Two vectors are said to be the negative of each other if their magnitudes are equal but directions are opposite.

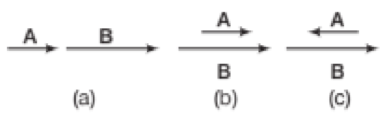
The negative vector of **A** is represented as −**A**.



Negative vector

(vi) Collinear Vectors

The two or more vectors are said to be collinear, when they act along the same lines or parallel lines. e.g. Tug of war.

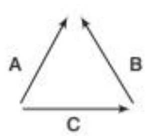


Collinear vectors

If **A** and **B** are two collinear vectors, then they can be represented along a line in the same direction [Fig. a] or along the parallel lines in same direction [Fig. b] or along parallel lines in opposite direction [Fig. c]. Two collinear vectors having the same directions are called **parallel vectors**. In this case, the angle between those two vectors will be zero. Similarly, the two collinear vectors having the opposite directions are called **anti-parallel vectors**. In this case, the angle between those two vectors will be 180°.

(vii) Coplanar Vectors

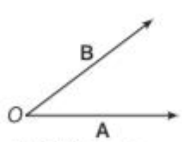
The vectors lying in the same plane are called **coplanar vectors**. Three vectors **A**, **B** and **C** are lying in the same plane of paper as shown in figure, hence they are coplanar vectors.



Coplanar vectors

(viii) Co-initial Vectors

The vectors which have the same initial point are called co-initial vectors.



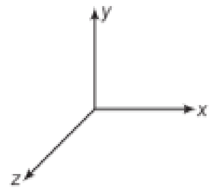
Co-initial vectors

Two vectors **A** and **B** have been drawn from the common initial point *O*. Therefore, **A** and **B** are called **co-initial vectors**.

(ix) Orthogonal Unit Vectors

If two or three unit vectors are perpendicular to each other, they are known as **orthogonal unit vectors**.

The unit vectors along *x*-axis, *y*-axis and *z*-axis are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. These are orthogonal unit vectors.



Orthogonal unit vectors

$$\hat{\mathbf{i}} = \frac{\mathbf{x}}{x} \Rightarrow \mathbf{x} = x\,\hat{\mathbf{i}} \; ; \quad \hat{\mathbf{j}} = \frac{\mathbf{y}}{y} \Rightarrow \mathbf{y} = y\,\hat{\mathbf{j}}$$

$$\hat{\mathbf{k}} = \frac{\mathbf{z}}{z} \Rightarrow \mathbf{z} = z\,\hat{\mathbf{k}}$$

(x) Localised Vectors

Those vectors whose initial point is fixed are known as **localised vectors**. e.g. Position vector of a particle (initial point lies at the origin).

(xi) Non-localised Vectors

Those vectors whose initial point is not fixed are known as **non-localised vectors.** e.g. Velocity vector of a particle moving along a straight line.

POSITION AND DISPLACEMENT VECTORS

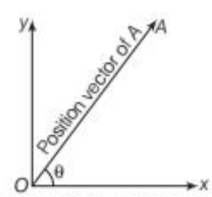
Position Vector

A vector which gives position of an object with reference to the origin of a coordinate system is called **position vector**. It is represented by a symbol **r**.

Consider the motion of an object in *xy*-plane with origin at *O*. Suppose an object is at point *A* at any instant *t*.

Then, **OA** is the position vector of the object at point *A*.

i.e. **OA** = **r**



Position vector representation

The position vector provides two informations such as

(i) It tells us about the minimum distance of an object from the origin O.

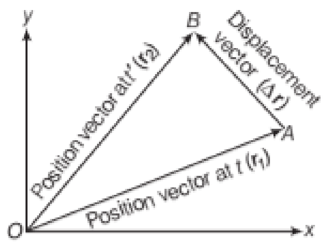
(ii) It tells us about the direction of the object w.r.t. origin.

Displacement Vector

The vector which tells how much and in which direction an object has changed its position in a given interval of time is called **displacement vector.**

Displacement vector is the straight line joining the initial and final positions and does not depend on the actual path undertaken by the object between the two positions.

Consider an object moving in the xy-plane. Suppose it is at point A at any instant t and at point B at any later instant t'. Then, vector **AB** is the displacement vector of the object in time t to t'.



Displacement vector representation

If the coordinates of points A and B are (x_1, y_1) and (x_2, y_2), then the position vector of the object at point A, $\mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$ and the position vector of the object at point B, $\mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$

∴ The displacement vector for AB can be given as

$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\boxed{\text{Displacement vector, } \Delta\mathbf{r} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}}}$$

(i) Magnitude of the displacement vector is given by

$$|\Delta\mathbf{r}| = \Delta r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of displacement is either less or equal to the path length of an object between two points.

(ii) Magnitude of vectors for three-dimensional is given by

$$\Delta\mathbf{r} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

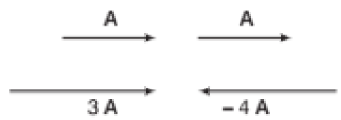
MULTIPLICATION OF A VECTOR BY A REAL NUMBER (OR SCALAR)

When we multiply a vector **A** by a real number λ, then we get a new vector along the direction of vector **A**. Its magnitude becomes λ times the magnitude of the given vector.

Similarly, if we multiply vector **A** with a negative real number $-\lambda$, then we get a vector whose magnitude is λ times the magnitude of vector **A** but direction is opposite to that of vector **A**.

Hence, $\boxed{\lambda(\mathbf{A}) = \lambda\mathbf{A}}$ and $\boxed{-\lambda(\mathbf{A}) = -\lambda\mathbf{A}}$

e.g. (i) Consider a vector **A** is multiplied by a real number $\lambda = 3$ or -4, we get 3**A** or -4 **A**



Multiplication of a vector by a real number

(ii) If we multiply a constant velocity vector by time, we will get a displacement vector in the direction of velocity vector.

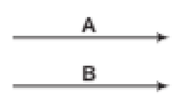
RESULTANT VECTOR

The resultant vector of two or more vectors is defined as the single vector which produces the same effect as two or more vectors (given vectors) combinedly produces.

There are two cases

Case I **When two vectors are acting in the same direction.**

Consider the vectors **A** and **B** are acting in the same direction as shown below

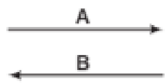


Then, the resultant of these two vectors is given by a vector having direction as same as that of **A** or **B** and the magnitude of the resultant vector will be equal to the sum of respective vectors i.e. $(A + B)$.

Thus, $\boxed{\text{Resultant vector, } \mathbf{R} = \mathbf{A} + \mathbf{B}}$

Case II When two vectors are acting in mutually opposite directions.

Consider the vectors **A** and **B** are acting in mutually opposite direction as shown below.



Then, the resultant of these two vectors is given by a vector having direction same as that of vector with larger magnitude. The magnitude of the resultant vector will be equal to $|A - B|$.

Thus, $\boxed{\text{Resultant vector, } \mathbf{R} = \mathbf{A} - \mathbf{B}}$

(i) If **B** > **A**, then direction of **R** is along **B**.

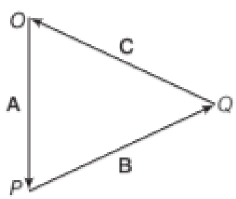
(ii) If **A** > **B**, then direction of **R** is along **A**.

Conditions for Zero Resultant Vector

If three vectors acting on a point object at the same time are represented in magnitude and direction by the three sides of a triangle taken in the same order, their resultant is zero. The object is said to be in equilibrium.

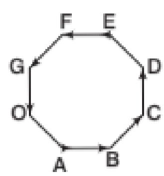
Consider the three vectors **A**, **B** and **C** acting on an object at the same time represented by **OP**, **PQ** and **QO**, respectively.

Then, $$\frac{\mathbf{A}}{\mathbf{OP}} = \frac{\mathbf{B}}{\mathbf{PQ}} = \frac{\mathbf{C}}{\mathbf{QO}}$$



Vectors **A**, **B** and **C** acting along **OP**, **PQ** and **QO** respectively

Similarly, if number of vectors acting on an object at the same time are represented in magnitude and direction by the various sides of a closed polygon taken in the same order, their resultant vector is zero and the object will be in equilibrium.



Vectors represented by a closed polygon

Resultant vector, **R** = **OA** + **AB** + **BC** + **CD** + **DE** + **EF** + **FG** + **GO** = 0

> **Conditions for Equilibrium of an Object**
> - There is no linear motion of the object i.e. the resultant force on the object is zero.
> - There is no rotational motion of the object i.e. the torque due to forces on the object is zero.
> - There is minimum potential energy of the object for stable equilibrium.

ADDITION OF VECTORS (GRAPHICAL METHOD)

Two vectors can be added if both of them are of same nature e.g. A displacement vector cannot be added to a force but can be added to displacement vector only.

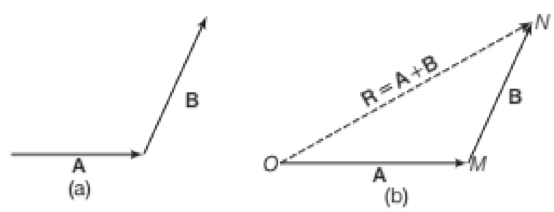
Graphical method of addition of vectors helps us in visualising the vectors and the resultant vectors.

This method contains following laws

1. Triangle Law of Vector Addition

This law states that **if two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order**, then their resultant is represented completely, both in magnitude and direction, by the third side of the triangle taken in the opposite order.

Consider two vectors **A** and **B** that lie in a plane as shown in Fig. (a). Draw a vector **OM** equal and parallel to vector **A** as shown in Fig. (b). From head of **OM**, draw a vector **MN** equal and parallel to vector **B**. Then, the resultant vector is given by **ON** which joins the tail of **A** and head of **B**.



Triangle law of vector addition

The resultant of **A** and **B** is **ON** = **OM** + **MN**

or $\boxed{\text{Resultant vector, } \mathbf{R} = \mathbf{A} + \mathbf{B}}$

EXAMPLE |1| Displacement of Vectors

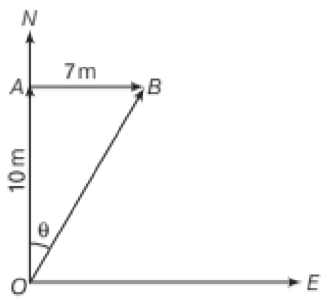
A boy travels 10 m due to North and then 7 m due to East. Find the displacement and direction of the body.

Sol. Let the boy start moving from point O as shown in figure.

where, **OA** = 10 m, due North

AB = 7 m, due East

According to triangle law of vector addition, **OB** is the resultant displacement.



The magnitude of the resultant displacement,

$$|\mathbf{OB}| = OB = \sqrt{(OA)^2 + (AB)^2} = \sqrt{(10)^2 + (7)^2}$$

$$= \sqrt{100 + 49} = 12.21 \text{ m}$$

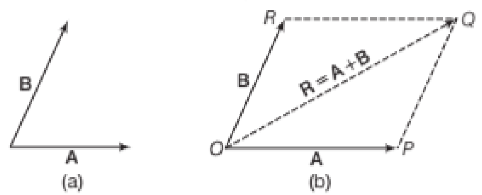
Since, the resultant displacement makes an angle θ with the North direction. Then,

$$\theta = \tan^{-1}\left(\frac{AB}{OA}\right) = \tan^{-1}\left(\frac{7}{10}\right) = 35°$$

2. Parallelogram Law of Vector Addition

This law states that **if two vectors acting on a particle at the same time are represented in magnitude and direction by two adjacent sides of a parallelogram drawn from a point** their resultant vector is represented in magnitude and direction by the diagonal of the parallelogram drawn from the same point.

Consider two vectors **A** and **B** that lie in a plane as shown in Fig. (a). From a common point O, draw a vector **OP** equal and parallel to **A** and vector **OR** equal and parallel to **B**. Complete the parallelogram **OPQR** as shown in Fig. (b). Then, the resultant vector is given by **OQ**.



Parallelogram law of vector addition

According to parallelogram law of vector addition,

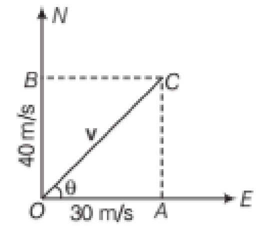
$$\mathbf{OQ} = \mathbf{OP} + \mathbf{OR}$$

or $\boxed{\text{Resultant vector, } \mathbf{R} = \mathbf{A} + \mathbf{B}}$

EXAMPLE |2| Velocities in Different Directions

A body is simultaneously given two velocities of 30 m/s due East and 40 m/s due North, respectively. Find the resultant velocity.

Sol. Let the body be starting from point O as shown.



$\mathbf{v}_A = 30 \text{ m/s}$, due East; $\mathbf{v}_B = 40 \text{ m/s}$, due North

According to parallelogram law, **OC** is the resultant velocity. Its magnitude is given by

$$\mathbf{v} = \sqrt{v_A^{\,2} + v_B^{\,2}}$$

$$= \sqrt{(30)^2 + (40)^2}$$

$$= \sqrt{900 + 1600}$$

$$= 50 \text{ m/s}$$

Since, the resultant velocity **v** makes an angle θ with the East direction. Then, $\theta = \tan^{-1}\frac{CA}{OA}$

$$= \tan^{-1}\left(\frac{40}{30}\right) = 53°8'$$

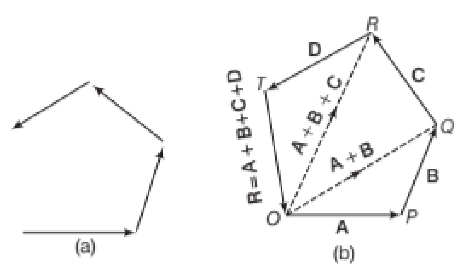
3. Polygon Law of Vector Addition

This law states that **when the number of vectors are represented in both magnitude and direction by the sides of an open polygon taken in an order,** then their resultant is represented in both magnitude and direction by the closing side of the polygon taken in opposite order.

Consider four vectors **A**, **B**, **C** and **D** be acting in different directions that lie in a plane as shown in Fig. (a). Draw a vector **OP** parallel and equal to vector **A**. Move vectors **B**, **C** and **D** parallel to themselves, so that the tail of **B** touches the head of **A**, the tail of **C** touches the head of **B** and the tail of **D** touches the head of **C** as shown in Fig. (b).

According to the polygon law of vector addition, the closing side OT of the polygon taken in the reverse order represent the resultant **R**.

Thus, $\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}$



Polygon law of vector addition

Properties of Addition of Vectors

(i) It follows commutative law, i.e. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

(ii) It follows associative law,

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$

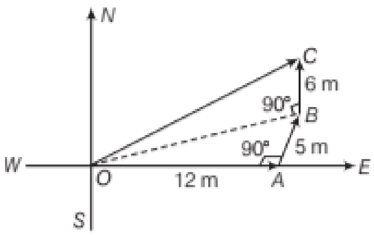
(iii) It obeys distributive law, $\lambda(\mathbf{A}+\mathbf{B})=\lambda\mathbf{A}+\lambda\mathbf{B}$

(iv) $\mathbf{A}+0=\mathbf{A}$

EXAMPLE |3| Displacement in Different Directions

A particle has a displacement of 12 m towards East and 5 m towards the North and then 6 m vertically upwards. Find the magnitude of the sum of these displacements.

Sol. Suppose initially the particle is at origin O. Then, its displacement vectors are



$$OA=12\text{m},\ AB=5\text{m},\ BC=6\text{ m}$$

According to polygon law of vector addition, **OC** is the resultant displacement.

$\therefore$ From $\Delta\,OAB$, have

$$\text{OB}=\sqrt{OA^2+AB^2}=\sqrt{(12)^2+(5)^2}=13\text{ m}$$

Again from $\Delta\,OBC$, we have

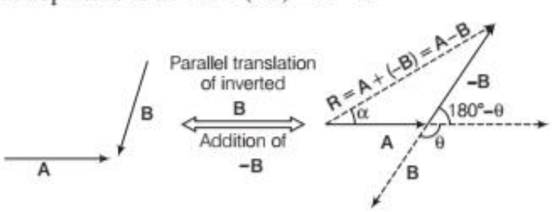
$$\text{OC}=\sqrt{OB^2+BC^2}=\sqrt{(13)^2+(6)^2}=\sqrt{205}=14.32\text{ m}$$

SUBTRACTION OF TWO VECTORS (GRAPHICAL METHOD)

If a vector **B** is to be subtracted from vector **A**, then we have to invert the vector **B** and then add it with vector **A** according to laws of addition of two vectors.

Hence, the subtraction of vector **B** from a vector **A** is defined as the addition of vector (−**B**) (i.e. negative of vector **B**) to vector **A**.

It is expressed as $\mathbf{R}=\mathbf{A}+(-\mathbf{B})=\mathbf{A}-\mathbf{B}$



Graphical method of subtraction of two vectors

Properties of Subtraction of Vectors

(i) Subtraction of vectors does not follow commutative law, i.e. $\mathbf{A}-\mathbf{B}\neq\mathbf{B}-\mathbf{A}$

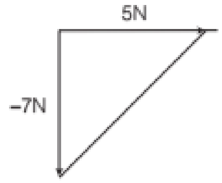
(ii) It does not follow associative law, i.e.

$$\mathbf{A}-(\mathbf{B}-\mathbf{C})\neq(\mathbf{A}-\mathbf{B})-\mathbf{C}$$

(iii) It follows distributive law, $\lambda(\mathbf{A}-\mathbf{B})=\lambda\mathbf{A}-\lambda\mathbf{B}$

EXAMPLE |4| Forces Act in Different Directions

Two forces of 5 N towards East and −7 N towards South acts on a particle. Find the resultant force.



Sol. The two forces be $A=5$ N and $B=-7$ N

Then, the magnitude of the resultant force is given by

$$F=\sqrt{A^2+(-B)^2}=\sqrt{5^2+(-7)^2}$$

$$F=\sqrt{25+49}=\sqrt{74}=8.6\text{ N}$$

RESOLUTION OF VECTORS IN PLANE (IN TWO DIMENSIONS)

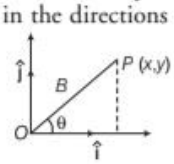
The process of splitting a single vector into two or more vectors in different directions which collectively produce the same effect as produced by the single vector alone is known as **resolution of a vector**.

The various vectors into which a single vector is splitted are known as **components of vectors**. Resolution of a vector into two component vectors along the directions of two given vectors is unique.

To understand the resolution of vector in the components vectors, let us discuss the vector as a combination of unit vectors.

Any vector **r** can be expressed as a linear combination of two unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ at right angle i.e. $\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}$

The vectors $x\hat{\mathbf{i}}$ and $y\hat{\mathbf{j}}$ are called the **perpendicular components** of **r**. The scalars x and y are called components or resolved parts of r in the directions of x-axis and y-axis.



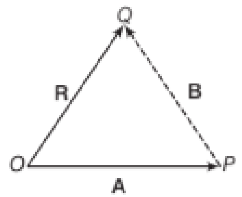
$\therefore$ Resultant vector, $\mathbf{r}=\sqrt{x^2+y^2}$

If θ is the inclination of **r** with *x*-axis, then

$$\text{Angle, } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Resolving a Vector into Two Component Vectors Along Given Directions

Now, draw **OQ** to represent the resultant vector **R** in magnitude and direction. From point *O*, draw a line **OP** parallel to the vector **A** and from point **P**, draw a line **PQ** parallel to vector **B**.

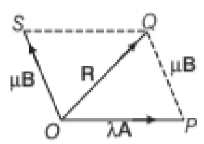


Then, these two lines intersect at point *P* as shown below.

From triangle law of vector addition, we have

$$\mathbf{OQ} = \mathbf{OP} + \mathbf{PQ}$$

But **OP** and **PQ** are two component vectors of **R** in the direction of **A** and **B** respectively. Let $\mathbf{OP} = \lambda\mathbf{A}$ and $\mathbf{PQ} = \mu\mathbf{B}$, where λ and μ are two real numbers. This is also illustrated in the figure as shown below.



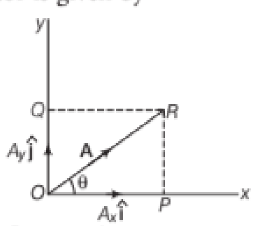
Now, the resultant vector becomes

$$\mathbf{R} = \lambda\mathbf{A} + \mu\mathbf{B}$$

Rectangular Components of a Vector in a Plane

When a vector in a plane is splitted into two component vectors at right angle to each other. Then, the component vectors are called **rectangular components** of that vector.

The resultant vector is given by



Rectangular components of **A**

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

where, $$\text{Magnitude of vector, } A = \sqrt{A_x^2 + A_y^2}$$

We can also find the angle (θ) between them.

From $$\tan\theta = \left(\frac{A_y}{A_x}\right)$$

$$\Rightarrow \quad \text{Angle, } \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

Where, A_y and A_x are the splitted vectors component of **A** in the direction of $\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}$, respectively.

EXAMPLE |5| Resultant of Two Vectors

The greatest and the least resultant of two forces acting at a point are 29 N and 5 N, respectively. If each force is increased by 3 N. Find the resultant of two new forces acting at right angle to each other.

Sol. Let *P* and *Q* be the two forces.

Greatest resultant, $R_1 = P + Q = 29$ N

Least resultant, $R_2 = P - Q = 5$ N

$$P + Q = 29 \quad \text{...(i)}$$

$$P - Q = 5 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$P = 17 \text{ N}, \; Q = 12 \text{ N}$$

When each force is increased by 3N, then

$$P' = P + 3 = 17 + 3 = 20 \text{ N}$$

$$\Rightarrow \quad Q' = 12 + 3 = 15 \text{ N}$$

The resultant of new forces is

$$R' = \sqrt{(20)^2 + (15)^2} = \sqrt{400 + 225} = 25 \text{ N}$$

Let the resultant **R**, makes an angle θ with **A**, then

$$\tan\theta = \frac{15}{20} = 0.75$$

$$\Rightarrow \quad \theta = \tan^{-1}(0.75) = 36°52'$$

Thus, the resultant of two new forces P' and Q' are 25 N and angle between them is 36°52′.

EXAMPLE |6| Person Walks in Different Directions

A person walks in the following pattern 3.1 km North, then 2.4 km West and finally 5.2 km South.

(i) Sketch the vector diagram that represents this motion.

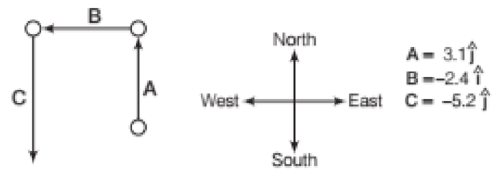
(ii) How far the person will be from its initial point?

(iii) In what direction would a bird fly in a straight line from the same starting point to the same final point?

Label the displacement vectors **A**, **B** and **C** (and denote the result of their vector sum as **r**) now, choose East as the $\hat{\mathbf{i}}$ direction (+*x* direction) and North as the $\hat{\mathbf{j}}$ direction (+*y* direction). All distances are understood to be in kilometres.

The vectors **B** and **C** are taken as negative because according to our assumption, person walks in $-x$ and $-y$ directions.

Sol. (i) The vector diagram representing the motion is shown below.



(ii) The final point is represented by

$$\mathbf{r} = \mathbf{A} + \mathbf{B} + \mathbf{C} = 3.1\hat{\mathbf{j}} - 2.4\hat{\mathbf{i}} - 5.2\hat{\mathbf{j}}$$

$$= -2.4\hat{\mathbf{i}} - 2.1\hat{\mathbf{j}}$$

whose magnitude is $|\mathbf{r}| = \sqrt{(-2.4)^2 + (-2.1)^2} = 3.2$ km

Hence, the person will be 3.2 km away from the initial point.

(iii) There are two possibilities for the angle.

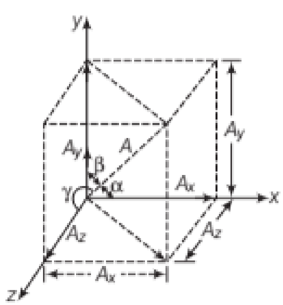
$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{-3.1}{-2.4}\right) = 41^\circ \text{ or } 221^\circ$$

$\because$ r is in the third quadrant, hence $\theta = 221^\circ$

Resolution of a Space Vector (In Three Dimensions)

Similarly, we can resolve a general vector **A** into three components along x, y and z-axes in three dimensions (i.e. space). Let α, β and γ are the angles between **A** and the x, y and z-axes, respectively as shown in figure.

Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be the unit vectors, along x, y and z-axes, respectively.



While resolving, we have,

$$A_x = A\cos\alpha,\ A_y = A\cos\beta,\ A_z = A\cos\gamma$$

$\therefore$ Resultant vector, $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$

Magnitude of vector **A** is $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ ...(i)

Position vector r is given by $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Remember that

$$\cos\alpha = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = l$$

$$\cos\beta = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = m$$

$$\cos\gamma = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = n$$

Here, l, m and n are known as **direction cosines of A**.

Putting the values of Ax, Ay and Az in Eq. (i), we get

$$A^2 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma$$

$$A^2 = A^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

or $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

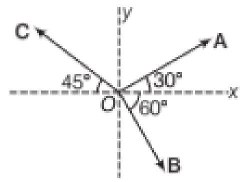
It means, sum of the squares of the direction cosines of a vector is always unity.

Note

The angles α, β and γ are angles in space. They are between pairs of lines, which are not coplanar.

EXAMPLE |7| Three Components of a Vector

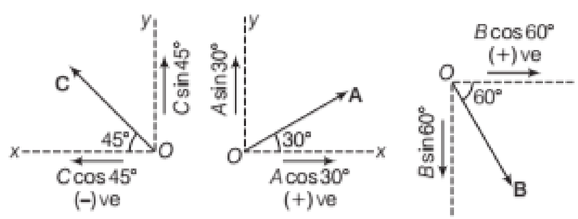
The figure shows three vectors **OA, OB** and **OC** which are equal in magnitude (say, F). Determine the direction of **OA** + **OB** – **OC**.



Sol. Given, $|\mathbf{OA}| = |\mathbf{OB}| = |\mathbf{OC}| = \mathbf{F}$

Angles are 30°, 45° and 60°.

Resolve all the vector components individually



$$\mathbf{R}_x = \mathbf{R}_{x_1} + \mathbf{R}_{x_2} + \mathbf{R}_{x_3}$$

Sum of vectors in x-direction (i.e. R_x) and sum of vactors in y-direction (i.e. R_y)

$$\mathbf{R}_x = A\cos 30^\circ + B\cos 60^\circ - C\cos 45^\circ$$

$$= \frac{F\sqrt{3}}{2} + \frac{F}{2} - \frac{F}{\sqrt{2}} \qquad [\because A = B = C = F]$$

$$= \frac{F}{2\sqrt{2}}(\sqrt{6} + \sqrt{2} - 2)$$

$$\mathbf{R}_y = A\sin 30^\circ + C\cos 45^\circ - B\sin 60^\circ$$

$$= \frac{F}{2} + \frac{F}{\sqrt{2}} - \frac{F\sqrt{3}}{2} = \frac{F}{2\sqrt{2}}(\sqrt{2} + 2 - \sqrt{6})$$

Determination of magnitude,

$$R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{\left[\frac{F}{2\sqrt{2}}(\sqrt{6}+\sqrt{2}-2)\right]^2 + \left[\frac{F}{2\sqrt{2}}(\sqrt{2}+2-\sqrt{6})\right]^2}$$

$$= \sqrt{F^2(0.435) + F^2(0.116)}$$

$$= \sqrt{F^2(0.550)} = F\sqrt{0.550}$$

$$\Rightarrow R = 0.74\ F$$

Determination of direction

$$\tan\theta = \frac{R_y}{R_x} = \frac{\frac{F}{2\sqrt{2}}(\sqrt{2}+2-\sqrt{6})}{\frac{F}{2\sqrt{2}}(\sqrt{6}+\sqrt{2}-2)}$$

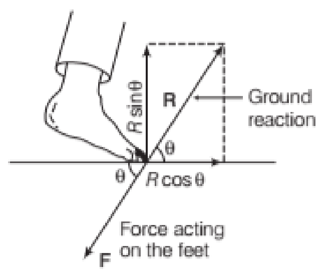
$$= \frac{0.97}{1.85} = 0.524$$

$$\theta = 27.65$$

This is the angle which **R** makes with x-axis.

EXAMPLE |8| Resolution of Forces

Can the walk of a man be an example of resolution of forces?



Sol. Walking of a man is an example of resolution of forces. A man while walking presses the ground with his feet backward by a force **F** at an angle θ with ground, in action.

The ground in reaction exerts an equal and opposite force **R** (= **F**) on the feet.

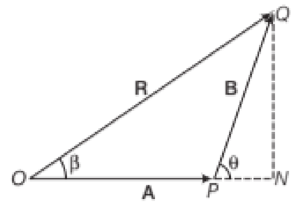
Its horizontal component $H = R\cos\theta$ enables the person to move forward while the vertical component $V = R\sin\theta$ balances his weight because R is resolved into two rectangular components.

Application of resolution of vector

It is easier to pull a lawn mower than to push.

Addition of Vectors (Analytical Method)

Consider two vectors **A** and **B** inclined at an angle θ be acting on a particle at the same time. Let they be represented in magnitude and direction by two sides **OP** and **PQ** of ΔOPQ, taken in the same order (take from above). Then, according to triangle law of vector addition, the resultant (**R**) is given by the closing side **OQ**, taken in opposite order.



Draw QN perpendicular to OP produced.

From ΔQNP, $\frac{PN}{PQ} = \cos\theta$

$$\Rightarrow \quad PN = PQ\cos\theta = B\cos\theta \quad [\because PQ = B] \quad \ldots\text{(i)}$$

and $\frac{QN}{PQ} = \sin\theta$

$$\Rightarrow \quad QN = PQ\sin\theta = B\sin\theta \quad \ldots\text{(ii)}$$

In right angled ΔONQ, we have

$$OQ^2 = QN^2 + NO^2 = QN^2 + (OP + PN)^2$$

or

$$R^2 = (B\sin\theta)^2 + (A + B\cos\theta)^2$$

$$= B^2\sin^2\theta + A^2 + B^2\cos^2\theta + 2\,AB\cos\theta$$

$$= A^2 + B^2 + 2A\,B\cos\theta$$

$$\Rightarrow \quad \boxed{\text{Resultant, } R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

This represents the magnitude of resultant vector **R**.

If the resultant vector **R** makes an angle (β) with the direction of vector **A**, then from right angle ΔQNO,

$$\tan\beta = \frac{QN}{ON} = \frac{QN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta}$$

or

$$\boxed{\text{Direction of resultant } \mathbf{R},\ \tan\beta = \frac{B\sin\theta}{A + B\cos\theta}}$$

Regarding the Magnitude of *R*

(i) When $\theta = 0°$, then $R = A + B$ (maximum).

(ii) When $\theta = 90°$, then $R = \sqrt{A^2 + B^2}$.

(iii) When $\theta = 180°$, then $R = A - B$ (minimum).

This can be extended to any number of vectors if vectors **a**, **b** and **c** are given, then

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

$$\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$$

$$\mathbf{c} = c_x\hat{\mathbf{i}} + c_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}}$$

where, $\mathbf{r}_x = a_x + b_x + c_x$, $\mathbf{r}_y = a_y + b_y + c_y$

$\mathbf{r}_z = a_z + b_z + c_z$

Determination of magnitude,

$$R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{\left[\frac{F}{2\sqrt{2}}(\sqrt{6}+\sqrt{2}-2)\right]^2 + \left[\frac{F}{2\sqrt{2}}(\sqrt{2}+2-\sqrt{6})\right]^2}$$

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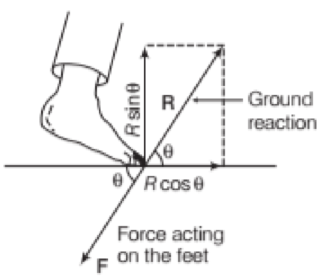

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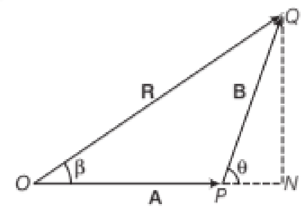

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$$= B^2\sin^2\theta + A^2 + B^2\cos^2\theta + 2AB\cos\theta$$

$$= A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow \quad \boxed{\text{Resultant, } R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

This represents the magnitude of resultant vector **R**.

If the resultant vector **R** makes an angle (β) with the direction of vector **A**, then from right angle ΔQNO,

$$\tan\beta = \frac{QN}{ON} = \frac{QN}{OP+PN} = \frac{B\sin\theta}{A + B\cos\theta}$$

or

$$\boxed{\text{Direction of resultant } \mathbf{R},\ \tan\beta = \frac{B\sin\theta}{A + B\cos\theta}}$$

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(iii) When $\theta = 180°$, then $R = A - B$ (minimum).

This can be extended to any number of vectors if vectors **a**, **b** and **c** are given, then

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

$$\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$$

$$\mathbf{c} = c_x\hat{\mathbf{i}} + c_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}}$$

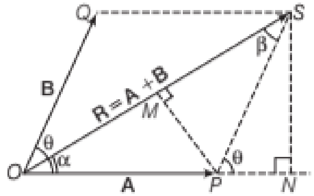
where, $\mathbf{r}_x = a_x + b_x + c_x$, $\mathbf{r}_y = a_y + b_y + c_y$

$$\mathbf{r}_z = a_z + b_z + c_z$$

EXAMPLE |9| Law of Sine and Law of Cosine

Find the magnitude and direction of the resultant of two vectors **A** and **B** in terms of their magnitudes and angle θ between them. [NCERT]

Sol. Let the vectors **OP** and **OQ** represent two vectors **A** and **B** which are at angle θ with each other as shown in figure.



As we know the magnitude of the resultant vector **R** is

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

This is known as **law of cosine**.

If the resultant vector **R** makes an angle α with the direction of vector **A**. Then,

In ΔOSN, $SN = OS\sin\alpha = R\sin\alpha$ and

In ΔPSN, $SN = PS\sin\theta = B\sin\theta$

Thus, $$\frac{R}{\sin\theta} = \frac{B}{\sin\alpha} \quad \text{...(i)}$$

Similarly, $$PM = A\sin\alpha = B\sin\beta$$

$$\frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{R}{\sin\theta} = \frac{A}{\sin\beta} = \frac{B}{\sin\alpha}$$

This is known as **law of sines.**

EXAMPLE |10| Resultant of Two Vectors

Two vectors **C** and **D** having magnitude 5 units and angle $\theta = 30°$ with each other. Find the resultant vector.

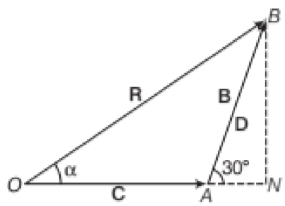
Sol. We have to find the resultant of given two vectors **C** and **D**. For the magnitude of resultant vector

$$R = |\mathbf{R}| = |\mathbf{C} + \mathbf{D}| = \sqrt{C^2 + D^2 + 2CD\cos\theta}$$

$$= \sqrt{(5)^2 + (5)^2 + 2\times5\times5\times\cos 30°}$$

$$= \sqrt{25 + 25 + 25\sqrt{3}} \quad \left(\because \cos 30° = \frac{\sqrt{3}}{2}\right)$$

$$= 5\sqrt{2+\sqrt{3}} \text{ units}$$



Now, for the direction of resultant vector

$$\tan\alpha = \frac{D\sin\theta}{C + D\cos\theta} = \frac{5\times\sin 30°}{5 + 5\times\cos 30°}$$

$$\tan\alpha = \frac{5\times1/2}{5+5\times\frac{\sqrt{3}}{2}} = \left(\frac{1}{2+\sqrt{3}}\right) = 2-\sqrt{3} = 0.268$$

$$\Rightarrow \quad \alpha = \tan^{-1}(0.268) \approx 15°$$

EXAMPLE |11| Resultant of Three Vectors

Three vectors are given by

$$\mathbf{A} = A_x\hat{\mathrm{i}} + A_y\hat{\mathrm{j}} + A_z\hat{\mathrm{k}}$$
$$\mathbf{B} = B_x\hat{\mathrm{i}} + B_y\hat{\mathrm{j}} + B_z\hat{\mathrm{k}}$$
$$\mathbf{C} = C_x\hat{\mathrm{i}} + C_y\hat{\mathrm{j}} + C_z\hat{\mathrm{k}}$$

Determine the magnitude of resultant vector.

Sol. Given, $$\mathbf{A} = A_x\hat{\mathrm{i}} + A_y\hat{\mathrm{j}} + A_z\hat{\mathrm{k}}$$
$$\mathbf{B} = B_x\hat{\mathrm{i}} + B_y\hat{\mathrm{j}} + B_z\hat{\mathrm{k}}$$
$$\mathbf{C} = C_x\hat{\mathrm{i}} + C_y\hat{\mathrm{j}} + C_z\hat{\mathrm{k}}$$

We can add these easily by analytical method.

The resultant of these three vectors are $\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C}$

$$A_x\hat{\mathrm{i}} + A_y\hat{\mathrm{j}} + A_z\hat{\mathrm{k}} + B_x\hat{\mathrm{i}} + B_y\hat{\mathrm{j}} + B_z\hat{\mathrm{k}} + C_x\hat{\mathrm{i}} + C_y\hat{\mathrm{j}} + C_z\hat{\mathrm{k}}$$

$$= (A_x + B_x + C_x)\hat{\mathrm{i}} + (A_y + B_y + C_y)\hat{\mathrm{j}} + (A_z + B_z + C_z)\hat{\mathrm{k}}$$

The magnitude of **R** is given by,

$$R = \sqrt{(A_x + B_x + C_x)^2 + (A_y + B_y + C_y)^2 + (A_z + B_z + C_z)^2}$$

EXAMPLE |12| Resultant Velocity of a Boat

A motor boat is racing towards North at 25 km/h and the water current in that region is 10 km/h in the direction of 60° East to South. Find the resultant velocity of the boat. [NCERT]

Sol. Given, velocity of motor boat, $v_b = 25$ km/h (towards North)

Velocity of water current, $v_c = 10$ km/h (towards East to South)

Direction, $\theta = 180° - 60° = 120°$

We know that,

$$v = \sqrt{v_b^2 + v_c^2 + 2v_b v_c\cos\theta}$$

$$v = \sqrt{(25)^2 + (10)^2 + 2\times25\times10\times\cos 120°}$$

$$v = \sqrt{625 + 100 + 500\left(-\frac{1}{2}\right)}, \quad v = 21.8 \text{ km/h}$$

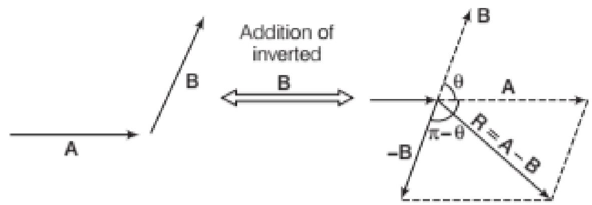
Suppose the resultant velocity v makes angle ϕ with the North direction. Then,

$$\tan\phi = \frac{v_c\sin 120°}{v_b + v_c\cos 120°} = \frac{10\times(\sqrt{3}/2)}{25 + 10\times(-1/2)} = 0.433$$

$$\therefore \quad \phi = \tan^{-1}(0.433) = 23.4°$$

Subtraction of Vectors
(Analytical Method)

There are two vectors **A** and **B** at an angle θ. If we have to subtract **B** from **A**, then first invert the vector **B** and then add with **A** as shown in figure.



Subtraction of vectors

The resultant vector is $\mathbf{R} = \mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}$

The magnitude of resultant in this case is

$$R = |\mathbf{R}| = \sqrt{A^2 + B^2 + 2AB\cos(\pi - \theta)}$$

$$\boxed{\text{Resultant, } R = \sqrt{A^2 + B^2 - 2AB\cos\theta}}$$

Here, θ = angle between **A** and **B**

Regarding the magnitude of R

(i) When $\theta = 0°$, then $R = A - B$ (minimum).

(ii) When $\theta = 90°$, then $R = \sqrt{A^2 + B^2}$.

(iii) When $\theta = 180°$, then $R = A + B$ (maximum).

EXAMPLE |13| Finding the Subtraction of Vectors

Two vectors of magnitude 3 units and 4 units are at angle 60 ° between them. Find the magnitude of their difference.

Sol. Let the vectors are **A** and **B**.

Given, $|\mathbf{A}| = 3$ unit, $|\mathbf{B}| = 4$ unit and $\theta = 60°$

The magnitude of resultant of **A** and **B**

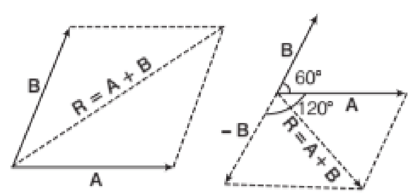
$$R = |\mathbf{R}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$
$$= \sqrt{3^2 + 4^2 - 2\times 3\times 4\cos 60°}$$
$$R = \sqrt{25 - 12} = \sqrt{13} = 3.61 \text{ units}$$

EXAMPLE |14| Sum and Difference Together

Consider vectors A and B having equal magnitude of 5 units and are inclined each other by 60°. Find the magnitude of sum and difference of these vectors.

Sol. Given, **A** = 5 units, **B** = 5 units, θ = 60°

A + **B** = ? and **A** − **B** = ?



The magnitude of the resultant vectors of the sum,

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$
$$= \sqrt{5^2 + 5^2 + 2\times 5\times 5\times \cos 60°} = 5\sqrt{3} \text{ unit}$$

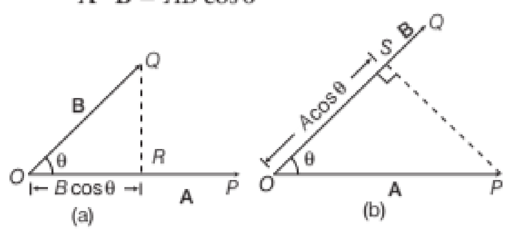
The magnitude of the resultant vector of the difference,

$$R = \sqrt{A^2 + (-B)^2 + 2AB\cos\theta}$$
$$R = \sqrt{5^2 + 5^2 + 2\times 5\times 5\cos 120°},$$
$$R = 5 \text{ unit}$$

DOT PRODUCT OR SCALAR PRODUCT

It is defined as the product of the magnitudes of vectors **A** and **B** and the cosine of the angle θ between them. It is represented by

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$$



(a) **B** cos θ is the projection of **B** onto **A**.
(b) **A** cos θ is the projection of **A** onto **B**.

Case I When the two vectors are parallel, then θ = 0°.
We have, $\mathbf{A}\cdot\mathbf{B} = AB\cos 0° = AB$

Case II When the two vectors are mutually perpendicular, then θ = 90°.
We have, $\mathbf{A}\cdot\mathbf{B} = AB\cos 90° = 0$

Case III When the two vectors are anti-parallel, then θ = 180°.
We have, $\mathbf{A}\cdot\mathbf{B} = AB\cos 180° = -AB$

Dot Product of Two Vectors in Terms of Their Components

It is defined as the product of the magnitude of one vector and the magnitude of the component of other vector in the direction of first vector.

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

then
$$\mathbf{a}\cdot\mathbf{b} = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})\cdot(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}})$$
$$= (a_1\cdot b_1)\hat{\mathbf{i}}^2 + (a_1\cdot b_2)\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} + (a_1\cdot b_3)\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}$$
$$+ (a_2\cdot b_1)\hat{\mathbf{j}}\cdot\hat{\mathbf{i}} + (a_2\cdot b_2)\hat{\mathbf{j}}^2 + (a_2\cdot b_3)\hat{\mathbf{j}}\cdot\hat{\mathbf{k}}$$
$$+ (a_3\cdot b_1)\hat{\mathbf{k}}\cdot\hat{\mathbf{i}} + (a_3\cdot b_2)\hat{\mathbf{k}}\cdot\hat{\mathbf{j}} + (a_3\cdot b_3)\hat{\mathbf{k}}^2.$$

$$= a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

where, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$$\hat{i}\cdot\hat{j} + \hat{i}\cdot\hat{k} + \hat{j}\cdot\hat{i} + \hat{k}\cdot\hat{i} = 0$$

Properties of Dot Product

(i) $\mathbf{a}\cdot\mathbf{a} = (\mathbf{a})^2$

(ii) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

(iii) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

(iv) $(\mathbf{c}\cdot\mathbf{a})\cdot\mathbf{b} = \mathbf{c}\cdot(\mathbf{a}\cdot\mathbf{b})$

(v) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

EXAMPLE |15| Dot Product of Two Vectors

Find the angle between the vectors

$$\mathbf{A} = \hat{i} - 2\hat{j} - \hat{k} \text{ and } \mathbf{B} = -\hat{i} + \hat{j} - 2\hat{k}$$

Sol. The angle between two vectors is included in the expression of dot product or scalar product.

$$\mathbf{A}\cdot\mathbf{B} = AB\cos Q \quad \text{...(i)}$$

The magnitude of **A** is given by

$$A = \sqrt{(1)^2 + (-2)^2 + (-1)^2} = \sqrt{6} \quad \text{...(ii)}$$

The magnitude of **B** is given by

$$B = \sqrt{(-1)^2 + (1)^2 + (-2)^2} = \sqrt{6} \quad \text{...(iii)}$$

We can separately evaluate the left side of Eq. (i) by writing the vectors in unit vector notation and using the distributive law.

$$\mathbf{A}\cdot\mathbf{B} = (\hat{i} + 2\hat{j} - \hat{k})\cdot(-\hat{i} + \hat{j} - 2\hat{k})$$

$$= (1\cdot -1)(\hat{i}\cdot\hat{i}) + (1\cdot 1)(\hat{i}\cdot\hat{j}) + (1\cdot -2)(\hat{i}\cdot\hat{k}) + (2\cdot -1)$$

$$(\hat{j}\cdot\hat{i}) + (2\cdot 1)(\hat{j}\cdot\hat{j}) + (2\cdot -2)(\hat{j}\cdot\hat{k}) + (-1\cdot -1)$$

$$(\hat{k}\cdot\hat{i}) + (-1\cdot 1)(\hat{k}\cdot\hat{j}) + (-1\cdot -2)(\hat{k}\cdot\hat{k})$$

$$\mathbf{A}\cdot\mathbf{B} = 3 \left[\begin{array}{l} \because\ \hat{i}\cdot\hat{i} = 1,\ \hat{i}\cdot\hat{j} = 0,\ \hat{i}\cdot\hat{k} = 0 \\ \hat{j}\cdot\hat{j} = 1,\ \hat{j}\cdot\hat{i} = 0,\ \hat{j}\cdot\hat{k} = 0 \\ \hat{k}\cdot\hat{k} = 1,\ \hat{k}\cdot\hat{i} = 0,\ \hat{k}\cdot\hat{j} = 0 \end{array}\right]$$

Substituting the values of **A** from Eq. (ii) and **B** from Eq. (iii) and $\mathbf{A}\cdot\mathbf{B} = 6$ in Eq. (i), we get

$$3 = (\sqrt{6})(\sqrt{6})\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

EXAMPLE |16| Application of Dot Product Method

A force of $(7\hat{i} + 6\hat{k})$ N makes a body move on a rough plane with a velocity of $(3\hat{i} + 4\hat{k})$ ms^{-1}. Calculate the power in W.

Sol. Using $\hat{k}\cdot\hat{k} = 1$, $\hat{i}\cdot\hat{j} = 0$

$$\therefore \text{ Power, } P = \mathbf{F}\cdot\mathbf{v} = (7\hat{i} + 6\hat{k})\cdot(3\hat{i} + 4\hat{k})$$

$$= 21 + 24 = 45 \text{ W} \qquad [\because\ \hat{i}\cdot\hat{k} = 0]$$

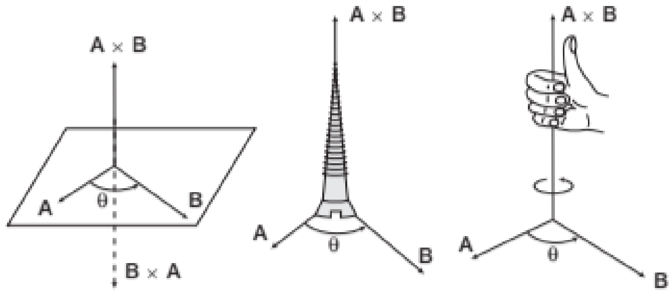
VECTOR PRODUCT OR CROSS PRODUCT

It is defined as the product of the magnitudes of vectors **A** and **B** and the sine of the angle θ between them.

It is represented as

Cross product of vectors **A** and **B**, $\mathbf{A}\times\mathbf{B} = AB\sin\theta\ \hat{\mathbf{n}}$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of $\hat{i}$.



Right hand rules for direction of vector product

Cross Product of Two Vectors in Terms of Their Components

If $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

where, $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

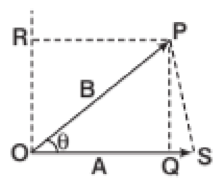
and $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$,

$$\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}$$

Case I Cross product of two parallel or anti-parallel vectors is zero.

Case II Cross product of two mutually perpendicular vector is equal to product of the magnitude of two vectors.

If **A** and **B** are two sides of Δ, then area of $\Delta = \frac{1}{2}|\mathbf{A}\times\mathbf{B}|$



Representation of vectors **A** and **B**

As, area of $\Delta = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} OS \times PQ$

$$\left(\begin{aligned} &\because \sin\theta = \frac{PQ}{OP} \\ &\Rightarrow PQ = OP \sin\theta = B\sin\theta \text{ and } OS = A \end{aligned} \right)$$

$$= \frac{1}{2} A \times B \sin\theta = \frac{1}{2} AB \sin\theta = \frac{1}{2} |\mathbf{A} \times \mathbf{B}|$$

Properties of Cross Product

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(ii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(iii) $(\mathbf{a} \times \mathbf{b}) + (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{d})$

(iv) $m\mathbf{a} \times \mathbf{b} = \mathbf{a} \times m\mathbf{b}$

(v) $(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$

(vi) $\mathbf{a} \times \mathbf{a} = 0$

(vii) $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$

(viii) $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2$

(ix) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\, \mathbf{c}$

EXAMPLE |17| Cross Product of Two Vectors

The vector **A** has a magnitude of 5 unit, **B** has a magnitude of 6 unit and the cross product **A** and **B** has the magnitude of 15 unit. Find the angle between **A** and **B**.

Sol. If the angle between **A** and **B** is θ, then cross product will have a magnitude,

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

$$\Rightarrow \quad 15 = 5 \times 6 \sin\theta$$

$$\sin\theta = \frac{1}{2} \Rightarrow \theta = 30^\circ$$

Representation of Unit Vectors in a Circle

Unit vectors along three axes of cartesian coordinate system (i.e. $\hat{i}, \hat{j}, \hat{k}$) can be represented on a circle in such a way that if we rotate our eyes in anti-clockwise product of two consecutive vector will produce the third unit vector.

e.g. $\hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$

and similarly for other possibilities.

> **Scalar Product of Vectors**
>
> The scalar product of vectors produce pseudo scalars, such as volume, power etc. The vector product of two vectors produces a pseudo vector. It is also called axial vector. The direction of this vector is perpendicular to the plane containing the multiple vectors.

TOPIC PRACTICE 1

OBJECTIVE Type Questions

1. Which one of the following statements is true? **[NCERT Exemplar]**

(a) A scalar quantity is the one that is conserved in a process

(b) A scalar quantity is the one that can never take negative values

(c) A scalar quantity is the one that does not vary from one point to another in space

(d) A scalar quantity has the same value for observers with different orientation of the axes

Sol. (d) A scalar quantity is independent of direction hence has the same value for observers with different orientations of the axes.

2. Consider the quantities, pressure, power, energy, impulse, gravitational potential, electrical charge, temperature, area. Out of these, the only vector quantity/ies is/are **[NCERT Exemplar]**

(a) impulse, pressure and area

(b) impulse

(c) area and gravitational potential

(d) impulse and pressure

Sol. (b) We know that, impulse, $J = F \cdot \Delta t = \Delta p$, where F is force, Δt is time duration and Δp is change in momentum. As Δp is a vector quantity, hence only impulse is also a vector quantity.

3. The relation between the vectors **A** and -2**A** is that,

(a) both have same magnitude

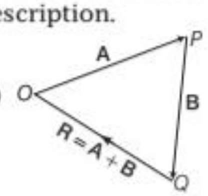
(b) both have same direction

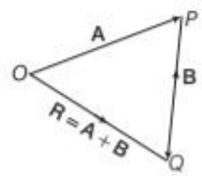
(c) they have opposite directions

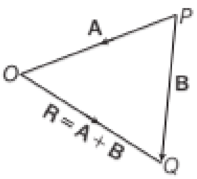
(d) None of the above

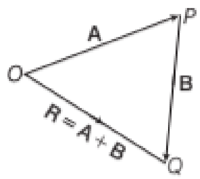
Sol. (c) Multiplying a vector **A** by a negative number λ gives a vector λ**A**, whose directions opposite to the direction of **A** and it's magnitude is $-\lambda$ times |**A**|.

4. A and B are two inclined vectors. R is their sum. Choose the correct figure for the given description.

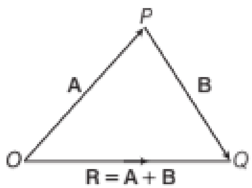
(a) 

(b) 

(c) 

(d) 

Sol. (d) Correct figure is



5. The component of a vector **r** along X-axis will have maximum value if **[NCERT Exemplar]**

(a) **r** is along positive y-axis

(b) **r** is along positive x-axis

(c) **r** makes an angle of 45° with the x-axis

(d) **r** is along negative y-axis

Sol. (b) Let **r** makes an angle θ with positive x-axis.

Component of **r** along X-axis

$$r_x = |\mathbf{r}| \cos\theta$$

$$(r_x)_{\text{maximum}} = |\mathbf{r}| (\cos\theta)_{\text{maximum}}$$

$$= |\mathbf{r}| \cos 0° = |\mathbf{r}|$$

($\because \cos\theta$ is maximum of $\theta = 0°$)

As $\theta = 0°$,

r is along positive x-axis.

VERY SHORT ANSWER Type Questions

6. Three vectors not lying in a plane can never end up to give a null vector. Is it true? **[NCERT]**

Sol. Yes, because they cannot be represented by the three sides of a triangle taken in the same order.

7. When do we say two vectors are orthogonal?

Sol. If the dot product of two vectors is zero, then the vectors are orthogonal.

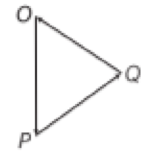
8. The total path length is always equal to the magnitude of the displacement vector of a particle. Why? **[NCERT]**

Sol. It is only true if the particle moves along a straight line in the same direction, otherwise the statement is false.

9. Under what condition, the three vectors give zero resultant?

Sol. If three vectors acting on a point object at the same time are represented in magnitude and direction by the three sides of a triangle taken in the same order, their resultant is zero.

The object is said to be in equilibrium.



10. What is the property of two vectors **A** and **B** such that **A** + **B** = **C** and $A + B = C$?

Sol. The two vectors are parallel and acting in the same direction i.e. $\theta = 0°$

11. What is the value of m in $\hat{\mathbf{i}} + m\hat{\mathbf{j}} + \hat{\mathbf{k}}$ to be unit vector?

Sol. For unit vector $|\hat{\mathbf{i}} + m\hat{\mathbf{j}} + \hat{\mathbf{k}}| = \sqrt{1 + m^2 + 1} = 1$

$$m^2 + 2 = 1$$

$$m^2 = -1 \Rightarrow m = \sqrt{-1}$$

$\therefore$ m is imaginary.

12. Two equal forces having their resultant equal to either. At what angle are they inclined?

Sol. $A = F, B = F, R = F, \theta = ?$

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

$$\Rightarrow \quad R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$F^2 = F^2 + F^2 + 2F^2\cos\theta$$

$$1 = 2(1 + \cos\theta)$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} - 1 = \frac{-1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$$

13. What is the angle made by vector, $\mathbf{A} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ with x-axis?

Sol. For the vector, $A_x = 2$, $A_y = 2$

We know that angle is given by

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{2}{2}\right)$$

$$\theta = \tan^{-1}(1) = 45°$$

14. The magnitude of vectors **A**, **B** and **C** are 12, 5 and 13 units respectively and **A** + **B** = **C**, find the angle between **A** and **B**.

Sol. We know that, $C^2 = A^2 + B^2$ or $13^2 = 12^2 + 5^2$

Thus, the angle between **A** and **B** is 90°.

15. What are the minimum number of forces which are numerically equal whose vector sum can be zero?

Sol. Two only, provided that they are acting in opposite directions.

16. Under what condition the three vectors cannot give zero resultant?

Sol. When the three vectors are not lying in one plane, they cannot produce zero resultant.

17. If $|\mathbf{A}+\mathbf{B}| = |\mathbf{A}-\mathbf{B}|$, what is the angle between $\mathbf{A}$ and $\mathbf{B}$?

Sol. $|A + B| = |A - B|$

$$\sqrt{A^2 + B^2 + 2\,AB\cos\theta} = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

$\Rightarrow \quad 4AB\cos\theta = 0 \Rightarrow \cos\theta = 0$

Hence, $\cos\theta = \cos 90^\circ$ or $\theta = \pi/2$

18. Can the scalar product of two vectors be negative?

Sol. Yes, it will be negative if the angle between the two vectors lies between 90° to 270°.

19. If $|\mathbf{A}\times\mathbf{B}| = \mathbf{A}\cdot\mathbf{B}$, what is the angle between $\mathbf{A}$ and $\mathbf{B}$?

Sol. As we know, $\mathbf{A}\times\mathbf{B} = AB\sin\theta$

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$$

According to the question

$$AB\sin\theta = AB\cos\theta$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ$$

20. What is the angle between A and B, if A and B denote the adjacent sides of a parallelogram drawn from a point and the area of the parallelogram is 1/2 AB?

Sol. Area of parallelogram $|\mathbf{A}\times\mathbf{B}| = AB\sin\theta = \frac{1}{2}AB$ [given]

$$\therefore \quad \sin\theta = \frac{1}{2} = \sin 30^\circ \text{ or } \theta = 30^\circ$$

21. If $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}\times\mathbf{B}|$, find the value of angle between A and B.

Sol. As $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}\times\mathbf{B}|$

$\therefore \quad AB\cos\theta = AB\sin\theta$ or $\tan\theta = 1$ or $\theta = \pi/4$

22. Can the walking on a road be an example of resolution of vectors?

Sol. Yes, when a man walks on the road, he presses the road along an oblique direction. The horizontal component of the reaction helps the man to walk on the road.

23. When the sum of the two vectors are maximum and minimum?

Sol. The sum of two vectors is maximum, when both the vectors are in the same direction and is minimum when they act in opposite direction.

As, $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

(i) For R to be maximum, $\cos\theta = +1$

$$\theta = 0^\circ$$

$$R_{max} = \sqrt{A^2 + B^2 + 2AB} = A + B$$

(ii) For R to be minimum,

$$\cos\theta = -1 \text{ or } \theta = 180^\circ$$

$$R_{min} = \sqrt{A^2 + B^2 + 2AB(-1)} = A - B$$

SHORT ANSWER Type Questions

24. We can order events in time and there is no sense of time, distinguishing past, present and future. Is time a vector?

Sol. We know that time always flows on and on i.e. from past to present and then to future.

Therefore, a direction can be assigned to time. Since, the direction of time is unique and it is unspecified or unstated. That is why, time cannot be a vector though it has a direction.

25. Two forces whose magnitudes are in the ratio 3 : 5 give a resultant of 28 N. If the angle of their inclination is 60°. Find the magnitude of each force.

Sol. Let A and B be the two forces.

Then, $A = 3x$, $B = 5x$, $R = 28$ N and $\theta = 60^\circ$

Thus, $\frac{A}{B} = \frac{3}{5}$

Now, $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow \quad 28 = \sqrt{9x^2 + 25x^2 + 30x^2\cos 60^\circ} = 7x \Rightarrow x = 4$

$\therefore$ Forces are $A = 12$ N and $B = 20$ N.

26. Suppose you have two forces F and F. How would you combine them in order to have resultant force of magnitudes?

(i) Zero (ii) F

Sol. (i) If they act at opposite direction, resultant is zero.

(ii) For the resultant to be F,

$$F^2 = F^2 + F^2 + 2F^2\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{-1}{2}$$

$$\Rightarrow \quad \theta = 120^\circ$$

27. Determine that vector which when added to the resultant of $\mathbf{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\mathbf{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$ gives unit vector along y-direction.

Sol. We are given, $\mathbf{A} = 3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

Thus, the resultant vector is given by

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (3\hat{i} - 5\hat{j} + 7\hat{k}) + (2\hat{i} + 4\hat{j} - 3\hat{k})$$

$$= 5\hat{i} - \hat{j} + 4\hat{k}$$

But the unit vector along y-direction $= \hat{j}$

$\therefore$ Required vector $= \hat{j} - (5\hat{i} - \hat{j} + 4\hat{k})$

$$= -5\hat{i} + 2\hat{j} - 4\hat{k}$$

28. Explain the property of two vectors **A** and **B** if $|\mathbf{A}+\mathbf{B}| = |\mathbf{A}-\mathbf{B}|$.

Sol. As we know that, $|\mathbf{A}+\mathbf{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

and $|\mathbf{A}-\mathbf{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

But as per question, we have

$$\sqrt{A^2 + B^2 + 2AB\cos\theta} = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

Squaring both sides, we have $(4\,AB\cos\theta) = 0$

$\Rightarrow \quad \cos\theta = 0 \quad$ or $\quad \theta = 90°$

Hence, the two vectors **A** and **B** are perpendicular to each other.

29. Two forces 5 kg-wt. and 10 kg-wt. are acting with an inclination of 120° between them. Find the angle when the resultant makes with 10 kg-wt.

Sol. Given, $A = 5$ kg-wt, $B = 10$ kg-wt, $\theta = 120°$ then $\beta = ?$

$$\tan\beta = \frac{B\sin\theta}{A + B\cos\theta} = \frac{10\sin 120°}{5 + 10\cos 120°} = \frac{5\sin 60°}{10 - 5\cos 60°}$$

$$= \frac{5\times\sqrt{3}/2}{10 - 5/2} = \frac{1}{\sqrt{3}} = \tan 30°$$

$\therefore \quad \beta = 30°$

30. The sum and difference of two vectors are perpendicular to each other. Prove that the vectors are equal in magnitude.

Sol. As the vectors $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$ are perpendicular to each other, therefore

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = 0$$

$$\mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B} = 0$$

or $\quad \mathbf{A} - \mathbf{B} = 0 \quad [\because \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}]$

$\Rightarrow \quad \mathbf{A} = \mathbf{B}$

31. The dot product of two vectors vanishes when vectors are orthogonal and has maximum value when vectors are parallel to each other. Explain.

Sol. We know that, $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, when vectors are orthogonal, $\theta = 90°$.

So, $\mathbf{A}\cdot\mathbf{B} = AB\cos 90° = 0$, when vectors are parallel, then, $\theta = 0°$.

So, $\quad \mathbf{A}\cdot\mathbf{B} = AB\cos 0° = AB$ (maximum)

32. The angle between vectors **A** and **B** is 60°. What is the ratio of $\mathbf{A}\cdot\mathbf{B}$ and $|\mathbf{A}\times\mathbf{B}|$?

Sol. The dot product, $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$ and cross product $|\mathbf{A}\times\mathbf{B}| = AB\sin\theta$.

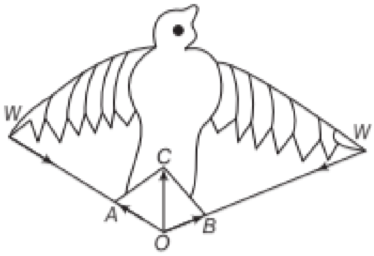
$\therefore$ Ratio is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \dfrac{AB\cos\theta}{AB\sin\theta} = \cot\theta$

$$= \cot 60° = \frac{1}{\sqrt{3}}$$

As, $\quad \theta = 60°, \cot 60° = \dfrac{1}{\sqrt{3}}$

33. Can a flight of a bird, an example of composition of vectors. Why?

Sol. Yes, the flight of a bird is an example of composition of vectors. As the bird flies, it strikes the air with its wings *W*, *W* along *WO*. According to Newton's third law of motion, air strikes the wings in opposite directions with the same force in reaction. The reactions are **OA** and **OB**. From law of parallelogram vectors, **OC** is the resultant of **OA** and **OB**. This resultant upwards force **OC** is responsible for the flight of the bird.



LONG ANSWER Type I Questions

34. Can you associate vectors with?

(i) The length of a wire bent into a loop

(ii) A plane area

(iii) A sphere **[NCERT]**

Sol. (i) We cannot associate a vector with the length of a wire bent into a loop.

(ii) We can associate a vector with a plane area.

Such a vector is called **area vector** and its direction is represented by outward drawn normal to the area.

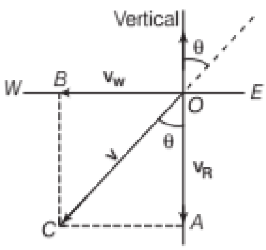
(iii) We cannot associate a vector with volume of sphere, however, a vector can be associated with the area of sphere.

35. On a certain day, rain was falling vertically with a speed of 35 m/s. A wind started blowing after sometime with a speed of 12 m/s in East to West direction. In which direction should a boy waiting at a bus stop hold his umbrella? **[NCERT]**

Sol. In figure

Velocity of rain, $\mathbf{V}_R = \mathbf{OA} = 35$ m/s, [vertically downward]

Velocity of wind, $\mathbf{V_W} = \mathbf{OB} = 12$ m/s, [East to West]



The magnitude of the resultant velocity is

$$v = \sqrt{(v_R)^2 + (v_W)^2}$$
$$= \sqrt{(35)^2 + (12)^2} = 37 \text{ m/s}$$

Let the resultant velocity, $\mathbf{v}(= \mathbf{OC})$ make an angle θ with the vertical. Then,

$$\tan\theta = \frac{AC}{OA} = \frac{v_W}{v_R} = \frac{12}{35} = 0.343$$

$\therefore \quad \theta = \tan^{-1}(0.343) \simeq 19°$

36. There are two displacement vectors, one of magnitude 3 m and the other of 4 m. How would the two vectors be added so that the magnitude of the resultant vector be (i) 7 m (ii) 1 m and (iii) 5 m?

Sol. The magnitude of resultant **R** of two vectors **A** and **B** is given by, $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$$= \sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \cos\theta}$$

(i) R is 7 m, if $\theta = 0°$
(ii) R is 1 m, if $\theta = 180°$
(iii) R is 5 m, if $\theta = 90°$

37. If unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are inclined at angle θ, then prove that $|\hat{\mathbf{a}} - \hat{\mathbf{b}}| = 2\sin\frac{\theta}{2}$. **[NCERT]**

Sol. For any vector a $\Rightarrow |\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$

$\therefore \quad |\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 = (\hat{\mathbf{a}} - \hat{\mathbf{b}}) \cdot (\hat{\mathbf{a}} - \hat{\mathbf{b}})$
$= \hat{\mathbf{a}} \cdot \hat{\mathbf{a}} - \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}} + \hat{\mathbf{b}} \cdot \hat{\mathbf{b}}$
$= 1 - 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} + 1 \qquad [\because \hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = 1 \times 1 \times \cos 0° = 1]$
$= 2 - 2 \times 1 \times 1 \times \cos\theta$
$= 2(1 - \cos\theta)$
$= 2.2\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2} \qquad [\because 1 - \cos 2\theta = 2\sin^2\theta]$

Hence, $|\hat{\mathbf{a}} - \hat{\mathbf{b}}| = 2\sin\frac{\theta}{2}$.

38. Show that vectors $A = 2\hat{i} - 3\hat{j} - \hat{k}$ and $B = -6\hat{i} + 9\hat{j} + 3\hat{k}$ are parallel.

Sol. The given vectors are

$$\mathbf{A} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

and $\quad \mathbf{B} = -6\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Then, the vectors are parallel, if $\mathbf{A} \times \mathbf{B} = 0$

$$\therefore \quad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(-9 + 9) - \hat{\mathbf{j}}(6 - 6) + \hat{\mathbf{k}}(18 - 18) = 0$$

But $|\mathbf{A} \times \mathbf{B}| = 0$

or $AB\sin\theta = 0 \qquad [\because \mathbf{A} \neq 0 \text{ and } \mathbf{B} \neq 0]$

$\therefore \quad \sin\theta = 0$ or $\theta = 0$

Hence, the vectors **A** and **B** are parallel.

LONG ANSWER Type II Questions

39. State with reasons, whether the following algebric operations with scalar and vector physical quantities are meaningful.
(i) Adding any two scalars.
(ii) Adding a scalar to a vector of the same dimensions.
(iii) Multiplying any vector by any scalar.
(iv) Multiplying any two scalars.
(v) Adding any two vectors.
(vi) Adding a component of a vector to the same vector. **[NCERT]**

Sol. (i) No, adding any two scalars is not meaningful because only the scalars of same dimensions i.e. having same unit can be added.

(ii) No, adding a scalar to a vector of the same dimensions is not meaningful because a scalar cannot be added to a vector.

(iii) Yes, multiplying any vector by any scalar is meaningful. When a vector is multiplied by a scalar, we get a vector, whose magnitude is equal to the product of magnitude of vector and the scalar and direction remains the same as the direction of the given vector.

e.g. A body of mass 4 kg is moving with a velocity 20 m/s towards East, then, product of velocity and mass gives the momentum of the body which is also a vector quantity.

$$\mathbf{p} = m\mathbf{v} = 4 \text{ kg} \times (20 \text{ m/s}) \text{ (East)} = 80 \text{ kg-m/s, East}$$

(iv) Yes, multiplying any two scalars is meaningful. Density ρ and volume V both are scalar quantities. When density is multiplied by volume, then we get $\rho \times V = m$, mass of the body, which is a scalar quantity.

(v) No, adding any two vectors is not meaningful because only vectors of same dimensions i.e. having same unit can be added.

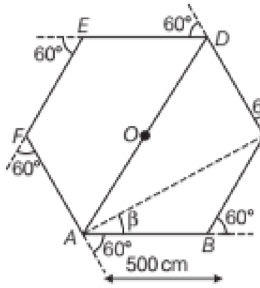
(vi) Yes, adding a component of a vector to the same vector is meaningful because both vectors are of same dimensions.

40. On an open ground, a motorist follows a track that turns to his left by an angle of 60° after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of the displacement with the total path length covered by the motorist in each case.

As motorist is taking turn to his left at an angle 60° after every 500 m, therefore, he is moving on a regular hexagon.

Sol. The distance after which motorist take a turn = 500 m
As motorist takes a turn at an angle of 60° each time, therefore motorist is moving on a regular hexagonal path. Let the motorist starts from point A and reaches at point D at the end of third turn and at initial point A at

the end of sixth turn and at point C at the end of eighth turn.



Displacement of the motorist at the third turn $= AD$

$$= AO + OD = 500 + 500 = 1000 \text{ m}$$

Total path length $= AB + BC + CD$

$$= 500 + 500 + 500 = 1500 \text{ m}$$

$$\therefore \frac{\text{Magnitude of displacement}}{\text{Total path length}} = \frac{1000}{1500} = \frac{2}{3} = 0.67$$

At the sixth turn motorist is at the starting point A.

$\therefore$ Displacement of the motorist at the sixth turn $= 0$

Total path length $= AB + BC + CD + DE + EF + FA$

$$= 500 + 500 + 500 + 500 + 500 + 500 = 3000 \text{ m}$$

$$\therefore \frac{\text{Magnitude of displacement}}{\text{Total path length}} = \frac{0}{3000} = 0$$

At the eighth turn, the motorist is at point C.

$\therefore$ Displacement of the motorist $= AC$

Using triangle law of vector addition,

$$AC = \sqrt{AB^2 + BC^2 + 2AB \cdot BC \cos 60^\circ}$$

$$= \sqrt{(500)^2 + (500)^2 + 2 \times 500 \times 500 \times \frac{1}{2}}$$

$$= \sqrt{3 \times (500)^2} = 500\sqrt{3} \text{ m}$$

$$AC = 500 \times 1.732 \text{ m} = 866 \text{ m}$$

If it is inclined at an angle β from the direction of **AB**,

$$\text{then } \tan\beta = \frac{500 \sin 60^\circ}{500 + 500\cos 60^\circ} = \frac{500 \times \frac{\sqrt{3}}{2}}{500 + 500 \times \frac{1}{2}}$$

$$= \frac{500 \times \frac{\sqrt{3}}{2}}{500\left(1 + \frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

or $\beta = 30^\circ$

$\therefore$ Displacement of the motorist at the end of eighth turn is 866 m making an angle 30° with the initial direction of motion.

Total path length $= 8 \times 500 = 4000$ m

$$\therefore \frac{\text{Magnitude of displacement}}{\text{Total path length}} = \frac{500\sqrt{3}}{4000} = \frac{\sqrt{3}}{8} = 0.22$$

41. Establish the following inequalities geometrically or otherwise.

(i) $|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$ (ii) $|\mathbf{A} + \mathbf{B}| \ge ||\mathbf{A}| - |\mathbf{B}||$

(iii) $|\mathbf{A} - \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$ (iv) $|\mathbf{A} - \mathbf{B}| \ge ||\mathbf{A}| - |\mathbf{B}||$

When does the equality sign above apply?

[NCERT]

Sol. Consider two vectors **A** and **B** be represented by the sides **OP** and **OQ** of a parallelogram *OPSQ*. According to parallelogram law of vector addition, $(\mathbf{A} + \mathbf{B})$ will be represented by **OS** as shown in figure.

Thus, $OP = |\mathbf{A}|$, $OQ = PS = |\mathbf{B}|$

and $OS = |\mathbf{A} + \mathbf{B}|$

(i) To prove $|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$

We know that the length of one side of a triangle is always less than the sum of the lengths of the other two sides. Hence from Δ *OPS*, we have

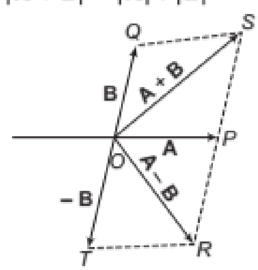
$$OS < OP + PS$$

$$\Rightarrow \quad OS < OP + OQ$$

or $\quad |\mathbf{A} + \mathbf{B}| < |\mathbf{A}| + |\mathbf{B}| \quad$ ...(i)

If the two vectors **A** and **B** are acting along the same straight line and in the same direction.

or $\quad |\mathbf{A} + \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| \quad$ ...(ii)



Combining the conditions mentioned in Eqs. (i) and (ii), we have $|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$

(ii) To prove $\quad |\mathbf{A} + \mathbf{B}| \ge ||\mathbf{A}| + |\mathbf{B}||$

From Δ *OPS*, we have

$$OS + PS > OP \text{ or } OS > |OP - PS|$$

or $\quad OS > |OP - OQ| \quad$ ... (iii)

$[\because PS = OQ]$

The modulus of $(OP - PS)$ has been taken because the LHS is always positive but the RHS may be negative if $OP < PS$. Thus, from Eq. (iii) we have

$$|\mathbf{A} + \mathbf{B}| > ||\mathbf{A}| - |\mathbf{B}|| \quad \text{...(iv)}$$

If the two vectors **A** and **B** are acting along a straight line in opposite directions, then

$$|\mathbf{A} + \mathbf{B}| = ||\mathbf{A}| - |\mathbf{B}|| \quad \text{...(v)}$$

Combining the conditions mentioned in Eqs. (iv) and (v) we get $\quad |\mathbf{A} + \mathbf{B}| \ge ||\mathbf{A}| - |\mathbf{B}||$

(iii) To prove $\quad |\mathbf{A} - \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$

In figure, $\quad \mathbf{A} = (\mathbf{OP})$, $-\mathbf{B} = \mathbf{OT} = \mathbf{PR}$

and $\quad (\mathbf{A} - \mathbf{B}) = \mathbf{OR}$

From ΔORP, we note that $OR < OP + PR$.

or $|\mathbf{A} - \mathbf{B}| < |\mathbf{A}| + |-\mathbf{B}|$

or $|\mathbf{A} - \mathbf{B}| < |\mathbf{A}| + |\mathbf{B}|$ ...(vi)

If the two vectors are acting along a straight line. But in the opposite direction, then

$$|\mathbf{A} - \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| \quad \text{...(vii)}$$

Combining the conditions mentioned in Eqs. (vi) and (vii), we get $|\mathbf{A} - \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$

(iv) To prove $|\mathbf{A} - \mathbf{B}| \ge \left||\mathbf{A}| + |\mathbf{B}|\right|$

In figure from Δ OPR, we note that [$\because OT = PR$]

$OR + PR > OP$ or $OR > |OP - PR|$

or $OR > |OP - OT|$ ...(viii)

The modulus of $(OP - OT)$ has been taken because LHS is positive and RHS may be negative if $OP < OT$.

From Eq. (viii) , we have

$$|\mathbf{A} - \mathbf{B}| > \left||\mathbf{A}| - |\mathbf{B}|\right| \quad \text{...(ix)}$$

If the two vectors $\mathbf{A}$ and $\mathbf{B}$ are acting along the same straight line in the same direction, then

$$|\mathbf{A} - \mathbf{B}| = |\mathbf{A}| - |\mathbf{B}| \quad \text{...(x)}$$

Combining the conditions mentioned in Eqs. (ix) and (x), we get $|\mathbf{A} - \mathbf{B}| \ge |\mathbf{A}| - |\mathbf{B}|$

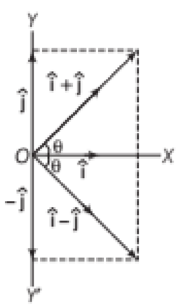
42. $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along X and Y-axes, respectively. What is the magnitude and direction of vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} - \hat{\mathbf{j}}$? What are the components of a vector $A = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ along the direction $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} - \hat{\mathbf{j}}$? (You may use graphical method).

The modulus of a vector $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$ is given by $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$. If vector is inclined at an angle θ, from x-axis, then $\tan\theta = \dfrac{A_y}{A_x}$

Sol. (i) Magnitude of $(\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

If vector $(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ makes an angle θ, with the x-axis, then

$$\tan\theta = \frac{A_y}{A_x} = \frac{1}{1} = 1 = \tan 45^\circ \text{ or } \theta = 45^\circ$$



(ii) Magnitude of

$$(\hat{\mathbf{i}} - \hat{\mathbf{j}}) = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

If vector $(\hat{\mathbf{i}} - \hat{\mathbf{j}})$ makes an angle θ, with x-axis, then

$$\tan\theta = \frac{A_y}{A_x} = \frac{(-1)}{1} = -1$$

$$= -\tan 45^\circ \Rightarrow \theta = -45^\circ \text{ with } \hat{\mathbf{i}}$$

Hence, vector $(\hat{\mathbf{i}} - \hat{\mathbf{j}})$ makes an angle of 45° from x-axis in negative direction.

(iii) To determine the component of $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ in the direction of $(\hat{\mathbf{i}} + \hat{\mathbf{j}})$.

Let $\mathbf{B} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})$

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta = (A\cos\theta) \cdot B$$

or $$A\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{B}$$

$\therefore$ Magnitude of the component of $\mathbf{A}$ in the direction of

$$\mathbf{B} = A\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{B} = \frac{(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}})}{\sqrt{(1)^2 + (1)^2}}$$

$$= \frac{2\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + 3\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}}{\sqrt{2}}$$

$$= \frac{2 + 3}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

(iv) Unit vector along $(\hat{\mathbf{i}} + \hat{\mathbf{j}})$, $\hat{\mathbf{n}} = \dfrac{(\hat{\mathbf{i}} + \hat{\mathbf{j}})}{|\hat{\mathbf{i}} + \hat{\mathbf{j}}|} = \dfrac{(\hat{\mathbf{i}} + \hat{\mathbf{j}})}{\sqrt{2}}$

Component of $\mathbf{A}$ along $(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

= Magnitude of the component of $\mathbf{A}$ along $(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \cdot \hat{\mathbf{n}}$

$$= \frac{5}{\sqrt{2}} \cdot \frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}})}{\sqrt{2}} = \frac{5}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

Magnitude of the component of $\mathbf{A}$ in the direction of

$$(\hat{\mathbf{i}} - \hat{\mathbf{j}}) = \frac{(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}})}{|\hat{\mathbf{i}} - \hat{\mathbf{j}}|} = \frac{2\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} - 3\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}}{\sqrt{(1)^2 + (-1)^2}}$$

$$= \frac{2 - 3}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Unit vector along $(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

$$\hat{\mathbf{n}} = \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}})}{|\hat{\mathbf{i}} - \hat{\mathbf{j}}|} = \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}})}{\sqrt{2}}$$

$\therefore$ Component of $\mathbf{A}$ along $(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

= Magnitude of the component of $\mathbf{A}$ along $(\mathbf{i} - \mathbf{j}) \cdot \hat{\mathbf{n}}$

$$= -\frac{1}{\sqrt{2}} \cdot \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}})}{\sqrt{2}} = -\frac{1}{2}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

ASSESS YOUR TOPICAL UNDERSTANDING

OBJECTIVE Type Questions

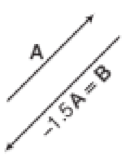
1. Choose the correct option(s).
   (a) To represent two-dimensional motion we need vectors
   (b) To represent one-dimensional motion we use positive and negative signs
   (c) To represent 3-dimensional motion we need vectors
   (d) All (a), (b) and (c)

2. $|\lambda \mathbf{A}| = \lambda |\mathbf{A}|$, if
   (a) $\lambda > 0$ (b) $\lambda < 0$
   (c) $\lambda = 0$ (d) $\lambda \neq 0$

3. If $\mathbf{A}$ is a vector with magnitude $|\mathbf{A}|$, then the unit vector $\hat{\mathbf{a}}$ in the direction of vector $\mathbf{A}$ is
   (a) $A\,\mathbf{A}$ (b) $\mathbf{A}\cdot\mathbf{A}$
   (c) $\mathbf{A}\times\mathbf{A}$ (d) $\dfrac{\mathbf{A}}{|\mathbf{A}|}$

4. Given, $|\mathbf{A}+\mathbf{B}| = P, |\mathbf{A}-\mathbf{B}| = Q$. The value of $P^2 + Q^2$ is
   (a) $2(A^2 + B^2)$ (b) $A^2 - B^2$
   (c) $A^2 + B^2$ (d) $2(A^2 - B^2)$

5. Choose the correct option regarding the given figure.



   (a) $\mathbf{B} = \mathbf{A}$ (b) $\mathbf{B} = -\mathbf{A}$
   (c) $|\mathbf{B}| = |\mathbf{A}|$ (d) $|\mathbf{B}| \neq |\mathbf{A}|$

Answer

1. (d)	2. (a)	3. (d)	4. (a)	5. (d)

VERY SHORT ANSWER Type Questions

6. How is a vector represented?
7. What is a zero vector? Explain the need of a zero vector.
8. State parallelogram law of vector addition. Show that the resultant of two vectors A and B inclined at an angle θ is
$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}.$$
9. Is the working of a sling based on the parallelogram law of vector addition, why?

SHORT ANSWER Type Questions

10. A vector $\mathbf{A}$ of magnitude A is turned through an angle α. Calculate the change in the magnitude of vector.

[**Ans.** $2A\sin\alpha/2$]

11. A hiker begins a trip by walking 25.0 km South-East from her base camp. On the second day she walks 40.0 km in direction 60.0° North to East, at which point she discovers a forest ranger's tower?
   (i) Determine the component of the hiker's displacements in the first and second days.
   (ii) Determine the component of the hiker's total displacement for the trip.
   (iii) Find the magnitude and direction of the displacement from base camp.

[**Ans.** (i) $\mathbf{A}_x = 17.7$ km, $\mathbf{A}_y = -17.7$ km, $\mathbf{B}_x = 20.0$ km, $\mathbf{B}_y = 34.6$ km
(ii) $\mathbf{R}_x = 37.7$ km, $\mathbf{R}_y = 16.9$ km,
(iii) $\mathbf{R} = 41.3$ km and $\theta = 24.1°$]

12. Is $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$ lie in the same plane?

LONG ANSWER Type I Questions

13. Is finite rotation of a vector?
14. Find the angle made by vector, $\mathbf{A} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ with X-axis.

[**Ans.** $\theta = 45°$]

15. It is easier to pull than to push a lawn roller. Why? [2]
16. Determine a unit vector which is perpendicular to both $\mathbf{A} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{B} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\left[\textbf{Ans. } \dfrac{3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{\sqrt{27}}\right]$

LONG ANSWER Type II Questions

17. A car is moving along a straight road with a uniform speed v. At a certain time, it is at a point Q-marked on the road. Suppose point O is taken as a fixed point, then show that $\mathbf{OQ}\times\mathbf{v}$ is independent of the position Q.
18. If $\mathbf{A}$ and $\mathbf{B}$ are two vectors such that $|\mathbf{A}\times\mathbf{B}| = \sqrt{3}\,\mathbf{A}\cdot\mathbf{B}$. Then,
   (i) Find the angle θ between $\mathbf{A}$ and $\mathbf{B}$
   (ii) Also, find the value of $|\mathbf{A}\times\mathbf{B}|$.

[**Ans.** (i) 60°, (ii) $\sqrt{A^2 + B^2 + AB}$]

|TOPIC 2|

Motion in a Plane

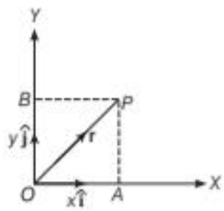
Here, we will discuss how to describe motion of an object in two dimensions using vectors.

POSITION, DISPLACEMENT AND VELOCITY VECTORS

Position Vector

A vector that extends from a reference point to the point at which particle is located is called **position vector.**

Let r be the position vector of a particle P located in a plane with reference to the origin O in xy-plane as shown in figure.



Representation of position vector

$$\mathbf{OP} = \mathbf{OA} + \mathbf{OB}$$

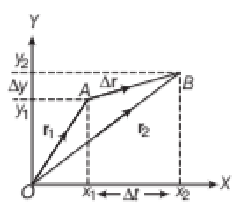
$$\text{Position vector, } \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

In three dimensions, the position vector is represented as

$$r = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Displacement

Consider a particle moving in xy-plane with a uniform velocity $\mathbf{v}$, Suppose O is the origin for measuring time and position of the particle. Let, the particle be at position A at time t_1 and at position B at time t_2, respectively. The position vectors are $\mathbf{OA} = \mathbf{r}_1$ and $\mathbf{OB} = \mathbf{r}_2$.



Representation of displacement vector

Then, the displacement of the particle in time interval $(t_2 - t_1)$ is **AB**. From triangle law of vector addition, we have

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB} \Rightarrow \mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

$$\mathbf{AB} = \mathbf{r}_2 - \mathbf{r}_1 \quad \ldots\text{(i)}$$

If the position coordinates of the particle at points A and B are (x_1, y_1) and (x_2, y_2), then

$$\therefore \quad \mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$$

$$\text{and} \quad \mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$$

Substituting the values of $\mathbf{r}_1$ and $\mathbf{r}_2$ in Eq. (i), we have

$$\mathbf{AB} = (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}) - (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}})$$

$$\text{Displacement, } \mathbf{AB} = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}}$$

Similarly, in three dimensions the displacement can be represented as

$$\Delta\mathbf{r} = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\,\hat{\mathbf{k}}$$

Velocity

The rate of change of displacement of an object in a particular direction is called its velocity.

It is of two types

Average Velocity

It is defined as the ratio of the displacement and the corresponding time interval.

$$\text{Thus, average velocity} = \frac{\text{displacement}}{\text{time taken}}$$

$$\text{Average velocity, } \mathbf{v}_{av} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$$

Velocity can be expressed in the component form as

$$\mathbf{v}_{av} = \frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}}$$

$$= v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$$

where, v_x and v_y are the components of velocity along x-direction and y-direction, respectively.

The magnitude of $\mathbf{v}_{av}$ is given by $\mathbf{v}_{av} = \sqrt{v_x^2 + v_y^2}$

and the direction of $\mathbf{v}_{av}$ is given by angle θ

$$\tan\theta = \frac{v_y}{v_x}$$

$$\Rightarrow \quad \text{Direction of average velocity, } \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

EXAMPLE |1| Average Velocity of Train

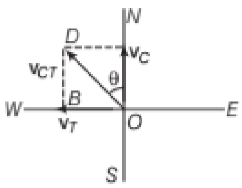
A train is moving with a velocity of 30 km/h due East and a car is moving with a velocity of 40 km/h due North. What is the velocity of car as appear to a passenger in the train?

Sol. Given, $\mathbf{v}_T = 30$ km/h due East

$\mathbf{v}_C = 40$ km/h due North

$\mathbf{v}_{CT} = ?, \theta = ?$

According to question,



In the figure,

$$\mathbf{v}_{CT} = OD = \sqrt{OB^2 + BD^2} = \sqrt{v_T^2 + v_C^2}$$

Then, $$\mathbf{v}_{CT} = \sqrt{(30)^2 + (40)^2} = \sqrt{900 + 1600} = 50 \text{ km/h}$$

$$\tan\theta = \frac{BD}{OD} = \frac{v_C}{v_T} \Rightarrow \tan\theta = \frac{30}{40}$$

$$\theta = \tan^{-1}\left(\frac{30}{40}\right) = 36°52' \text{ West of North}$$

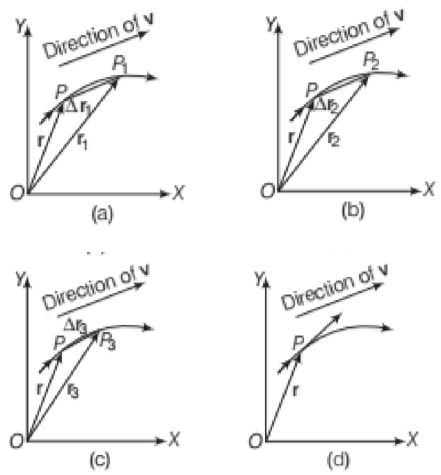
Instantaneous Velocity

The velocity at an instant of time (t) is known as **instantaneous velocity.**

The average velocity will become instantaneous, if Δt approaches to zero. The instantaneous velocity is expressed as

$$\text{Instantaneous velocity, } \mathbf{v}_i = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

The limiting process can be easily understood with the help of figure.



Limiting processes of instantaneous velocity

In the above figure, the curve represents the path of an object. The object is at point P on the path at time t. P_1, P_2 and P_3 are the positions of the object after time intervals $\Delta t_1, \Delta t_2$ and Δt_3, where $\Delta t_1 > \Delta t_2 > \Delta t_3$.

As the time interval Δt approaches zero, the average velocity approaches the velocity **v**. The direction of **v** is parallel to the line tangent to the path.

Note

The direction of instantaneous velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion.

EXAMPLE |2| Instantaneous Velocity of a Particle

A particle starts from origin at $t = 0$ with a velocity $5\hat{\mathbf{i}}$ m/s and moves in xy-plane under action of a force which produces a constant acceleration of $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$ m/s^2.

(i) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m?

(ii) What is the speed of the particle at this time?

[NCERT]

Sol. (i) Given, $v_0 = 5\hat{\mathbf{i}}$ m/s, $a = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ m/s^2

(i) $y(t) = ?$, $x(t) = 84$ m (ii) Speed $v = ?$

Then, $y(t) = v_0 t + \frac{1}{2}at^2$ and $v = \sqrt{v_x^2 + v_y^2}$

$$y(t) = 5\hat{\mathbf{i}}\,t + \frac{1}{2}(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})t^2$$

$$= \left(5t + \frac{3}{2}t^2\right)\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$$

On comparing, $x(t) = 5t + \frac{3}{2}t^2$

$\Rightarrow$ $y(t) = 1\,t^2$

(ii) The speed of the particle can be find by differentiating the position vector w.r.t. time.

If $x(t) = 84$ m, then $t = 6$ s

$\therefore$ $y(6) = 36$ m

$$v = \frac{dy}{dt} = \frac{d(5t + \frac{3}{2}t^2)\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}}{dt} = (5 + 3t)\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$$

At $t = 6$s, $v = 23\hat{\mathbf{i}} + 12\hat{\mathbf{j}}$

$\therefore$ $v = \sqrt{(23)^2 + (12)^2} = 26$ m/s

Acceleration

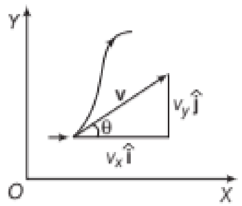
It is defined as the ratio of change in velocity and the corresponding time interval. It can be expressed as

$$\text{Acceleration, } \mathbf{a} = \frac{\text{Change in velocity}}{\text{Time taken}} = \frac{\Delta \mathbf{v}}{\Delta t}$$

$$\text{Acceleration, } \mathbf{a} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}$$

Average Acceleration

It is defined as the change in velocity ($\Delta \mathbf{v}$) divided by the corresponding time interval (Δt). It can be expressed as



Components of velocity

$$\text{Average acceleration, } \mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta v_x \hat{\mathbf{i}} + \Delta v_y \hat{\mathbf{j}}}{\Delta t}$$

$$= \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}}$$

$$\boxed{\text{Average acceleration, } \mathbf{a}_{av} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}}$$

Which is expressed in component form.

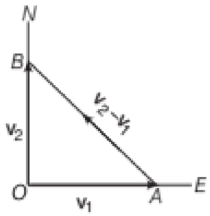
In terms of x and y, a_x and a_y can be expressed as

$$a_x = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \quad \text{and} \quad a_y = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

EXAMPLE |3| Average Acceleration of Particle

A particle is moving Eastwards with a velocity of 5 m/s in 10 second, the velocity changes to 5 m/s Northwards. Find the average accelaration of the particle in this time interval.

Sol.



According to triangle law of vector addition,

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$$

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{v}_2 - \mathbf{v}_1$$

$$= \text{Change in velocity}$$

$$\therefore \quad |\mathbf{v}_2 - \mathbf{v}_1| = \mathbf{AB} = \sqrt{(\mathbf{OA})^2 + (\mathbf{OB})^2}$$

$$= \sqrt{(5)^2 + (5)^2} = 5\sqrt{2} \text{ m/s}$$

$$\text{Hence, average acceleration} = \frac{|\mathbf{v}_2 - \mathbf{v}_1|}{t}$$

$$= \frac{5\sqrt{2}}{10} = \frac{1}{\sqrt{2}} \text{ m/s}^2$$

Along North-West direction.

Instantaneous Acceleration

It is defined as the limiting value of the average acceleration as the time interval approaches to zero.

It can be expressed as

$$\mathbf{a}_i = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

$$\boxed{\text{Instantaneous acceleration, } \mathbf{a}_i = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}}$$

where, $a_x = \dfrac{dv_x}{dt}, a_y = \dfrac{dv_y}{dt}$

The magnitude of instantaneous acceleration is given by

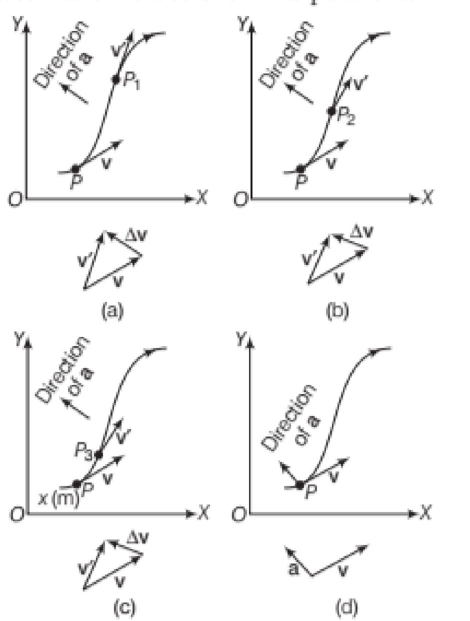
$$a_i = \sqrt{a_x^2 + a_y^2}$$

The limiting process can be easily understood with the help of figure.

The object is at point P at time t. P_1, P_2 and P_3 represent the positions of the object after time intervals $\Delta t_1, \Delta t_2$ and Δt_3, respectively. The time interval at the different positions in such way that $\Delta t_1 > \Delta t_2 > \Delta t_3$.

The velocity vectors at points P, P_1, P_2 and P_3 are also shown in figures.

In each case of Δt, the change in velocity $\Delta \mathbf{v}$ is obtained by using triangle law of vector addition. The direction of the average acceleration is also shown as parallel to $\Delta \mathbf{v}$.



Limiting processes of instantaneous acceleration

The average acceleration for three intervals (a) Δt_1, (b) Δt_2 and (c) $\Delta t_3, (\Delta t_1 > \Delta t_2 > \Delta t_3)$. (d) In the limit $\Delta t \to 0$, the average acceleration becomes the instantaneous acceleration called simply **acceleration**.

From the given figures, it can be evaluated as the time interval Δt decreases from (a) to (d), the direction of $\Delta \mathbf{v}$ and hence that of $\mathbf{a}$ changes. In Fig. (d), the time interval $\Delta t \to 0$, hence, the average acceleration becomes the instantaneous acceleration having direction as shown in figure.

Note

In two or three-dimensions, velocity and acceleration vectors can have any angle between 0° to 180° whereas in one dimension, the velocity and acceleration of an object are always along the same straight line (may be in same direction or in opposite direction).

EXAMPLE |4| Position Vector

The position of a particle is given by

$$\mathbf{r} = 3.0\, t\, \hat{\mathbf{i}} + 2.0\, t^2 \hat{\mathbf{j}} + 5.0\, \hat{\mathbf{k}}$$

where, t is in seconds and the coefficients have the proper units for r to be in metres.

(i) Find $v(t)$ and $a(t)$ of the particles.

(ii) Find the magnitude and direction of $v(t)$ at $t = 2.0$ s.

[NCERT]

Sol. Position of particle, $\mathbf{r} = 3.0\, t\, \hat{\mathbf{i}} + 2.0 t^2 \hat{\mathbf{j}} + 5.0\, \hat{\mathbf{k}}$

(i) $v(t) = ?$ and $a(t) = ?$

(ii) $v(t) = ?$, if $t = 1.0$ s, $\theta = ?$

(i) $\mathbf{v}(t) = \dfrac{dr}{dt} = \dfrac{d}{dt}(3.0\, t\, \hat{\mathbf{i}} + 2.0 t^2\, \hat{\mathbf{j}} + 5.0\, \hat{\mathbf{k}})$

$$= 3.0\, \hat{\mathbf{i}} + 4.0 t\, \hat{\mathbf{j}}$$

$$\mathbf{a}(t) = \frac{dv}{dt} = -4.0\hat{\mathbf{j}}\ \text{ms}^{-2}$$

(ii) At $t = 2.0$ s

$$\mathbf{v}(t) = 3.0\, \hat{\mathbf{i}} - 8.0\hat{\mathbf{j}}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(3.0)^2 + (-8)^2} = \sqrt{73} = 8.54\ \text{ms}^{-2}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \tan^{-1}\left(\frac{8}{3}\right) = -2.667 = -\tan 69.5°$$

$$\theta = 69.5° \text{ below } X\text{-axis}$$

Motion in a Plane with Uniform Velocity

A body is said to be moving with uniform velocity, if it suffers equal displacements in equal intervals of time, however small. Consider an object moving with uniform velocity $\mathbf{v}$ in xy-plane. Let $\mathbf{r}(0)$ and $\mathbf{r}(t)$ be its position vectors at $t = 0$ and $t = t$, respectively.

Then, $\mathbf{v} = \dfrac{\mathbf{r}(t) - \mathbf{r}(0)}{t - 0} \Rightarrow \mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}t$ ...(i)

In terms of rectangular coordinates, we get

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}, \quad v = \sqrt{v_x^2 + v_y^2}$$

$$\mathbf{r}(0) = x(0)\hat{\mathbf{i}} + y(0)\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$

On substituting these values in Eq. (i), we have

$$x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} = x(0)\hat{\mathbf{i}} + y(0)\hat{\mathbf{j}} + (v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}})t$$

$$x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} = [x(0) + v_x t]\hat{\mathbf{i}} + [y(0) + v_y t]\hat{\mathbf{j}} \quad \text{...(ii)}$$

By equating the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, we have

$$x(t) = x(0) + v_x t \quad \text{and} \quad y(t) = y(0) + v_y t$$

These two equations represent uniform motion along X-axis and Y-axis, respectively.

Eq. (ii) shows that the uniform motion in two dimensions can be expressed as the sum of two uniform motions along two mutually perpendicular directions.

Motion in a Plane with Constant Acceleration

A body is said to be moving with uniform acceleration, if its velocity vector suffers the same change in the same interval of time, however small.

Let an object is moving in xy-plane and its acceleration $\mathbf{a}$ is constant. At time $t = 0$, the velocity of an object be $\mathbf{v}_0$ (say) and $\mathbf{v}$ be the velocity at time t.

According to definition of average acceleration, we have

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v}_0}{t - 0} = \frac{\mathbf{v} - \mathbf{v}_0}{t}$$

$$\Rightarrow \quad \boxed{\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t}$$

In terms of rectangular components, we can express it as

$$v_x = v_{ox} + a_x t \text{ and } v_y = v_{oy} + a_y t$$

It can be concluded that each rectangular component of velocity of an object moving with uniform acceleration in a plane depends upon time as, if it were the velocity vector of one-dimensional uniformly accelerated motion.

Now, we can also find the position vector ($\mathbf{r}$). Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of the particle at time $t = 0$ and $t = t$ and their velocities at these instants be $\mathbf{v}_0$ and $\mathbf{v}$. Then, the average velocity is given by

$$\mathbf{v}_{av} = \frac{\mathbf{v}_0 + \mathbf{v}}{2}$$

Displacement is the product of average velocity and time interval. It is expressed as

$$\mathbf{r} - \mathbf{r}_0 = \left(\frac{\mathbf{v} + \mathbf{v}_0}{2}\right)t = \left[\frac{(\mathbf{v}_0 + \mathbf{a}t) + \mathbf{v}_0}{2}\right]t$$

$$\Rightarrow \quad \mathbf{r} - \mathbf{r}_0 = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

$$\Rightarrow \quad \boxed{\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2}$$

In terms of rectangular components, we have

$$x\hat{\mathbf{i}}+y\hat{\mathbf{j}}=x_0\hat{\mathbf{i}}+y_0\hat{\mathbf{j}}+(v_{0x}\hat{\mathbf{i}}+v_{0y}\hat{\mathbf{j}})t+\frac{1}{2}(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}})t^2$$

Now, equating the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$,

$$\boxed{x=x_0+v_{0x}t+\frac{1}{2}a_xt^2}$$

$$\boxed{y=y_0+v_{0y}t+\frac{1}{2}a_yt^2}$$

Note

Motion in a plane (two-dimensional motion) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions.

EXAMPLE |5| Particle Starts from Origin

A particle starts from origin at $t=0$ with a velocity $15\,\hat{\mathbf{i}}$ m/s and moves in xy-plane under the action of a force which produces a constant acceleration of $15\hat{\mathbf{i}}+10\hat{\mathbf{j}}$ m/s^2. Find the y-coordinate of the particle at the instant. Its x-coordinate is 125 m.

Sol. The position of the particle is given by

$$r(t)=v_0t+\frac{1}{2}at^2=15\hat{\mathbf{i}}t+\frac{1}{2}(15\hat{\mathbf{i}}+10\hat{\mathbf{j}})t^2$$

$$=(15t+7.5t^2)\hat{\mathbf{i}}+5\hat{\mathbf{j}}t^2$$

$\therefore$ $x(t)=15t+7.5t^2 \Rightarrow y(t)=5t^2$

If $x(t)=125$ m, $t=?$

$125=15t+7.5t^2 \Rightarrow 1.5t^2+3t-25=0$

$t=3.2$ s

$\therefore$ $y(t)=5\times(3.2)^2=51.2$ m

PROJECTILE MOTION

Projectile motion is a form of motion in which an object or particle is thrown with some initial velocity near the earth's surface and it moves along a curved path under the action of gravity alone. The path followed by a projectile is called its **trajectory**.

An object that is in flight after being thrown is called projectile.

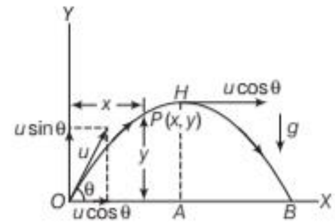
(i) A tennis ball or a baseball in a flight.
(ii) A bullet fired from a rifle.
(iii) A body dropped from the window of a moving train.
(iv) A jet of water flowing from a hole near the bottom of water tank.
(v) A javelin thrown by an athlete.

Assumptions before the Study of Projectile Motion

(i) There is no frictional resistance of air.
(ii) The effect due to rotation of earth and the curvature of the earth is negligible.
(iii) The acceleration due to gravity is constant both in magnitude and direction, at all points during the motion of projectile.

Mathematical Analysis of Projectile Motion

Let OX be a horizontal line on the ground, OY be a vertical line perpendicular to ground and O be the origin for XY-axes on a plane. Suppose an object is projected from point O with velocity (u), making an angle (θ) with the horizontal direction OX, such that $x_0=0$ and $y_0=0$ when $t=0$.



A projectile motion

While resolving velocity (u) into two components, we get (a) $u\cos\theta$ along OX and (b) $u\sin\theta$ along OY

As the horizontal component of velocity ($u\cos\theta$) is constant throughout the motion, so there is a constant acceleration and hence force is in the horizontal direction, if air resistance is assumed to be zero. As the vertical components of velocity ($u\sin\theta$) decreases continuously with height, from O to H, due to downward force of gravity and becomes zero.

At point H, the object has only horizontal component velocity ($u\cos\theta$). It attains a maximum height at AH. OB is the maximum horizontal range.

Note

The horizontal and vertical components of projectile motion was stated by Galileo.

Equation of Path of a Projectile

Suppose at any time t_1, the object reaches at point $P(x, y)$.

So, x = horizontal distance travelled by object in time t.

y = vertical distance travelled by object in time t.

Motion Along Horizontal Direction (OX)

The velocity of an object in horizontal direction i.e. OX is constant, so the acceleration a_x in horizontal direction is zero.

∴ Position of the object at time t along horizontal direction is given by $x = x_0 + u_x t + \frac{1}{2} a_x t^2$

But $x_0 = 0, u_x = u\cos\theta, a_x = 0$ and $t = t$

∴ $x = u\cos\theta\, t$

$$\Rightarrow \boxed{\text{Time, } t = \frac{x}{u\cos\theta}} \quad \ldots \text{(i)}$$

Motion Along Vertical Direction (OY)

The vertical velocity of the object is decreasing from O to H due to gravity, so acceleration a_y is g.

∴ Position of the object at any time t along the vertical direction i.e. OY is given by

$$y = y_0 + u_y t + \frac{1}{2} a_y t^2$$

But $y_0 = 0, u_y = u\sin\theta, a_y = -g$

and $t = t$

So, $y = u\sin\theta\, t + \frac{1}{2}(-g)t^2$

$$= u\sin\theta\, t - \frac{1}{2} g t^2 \quad \ldots \text{(ii)}$$

Substituting the value of t from Eq. (i) in Eq. (ii), we get

$$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2} g\left(\frac{x}{u\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{1}{2} g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\boxed{\text{Total vertical distance, } y = x\tan\theta - \left(\frac{1}{2}\frac{g}{u^2\cos^2\theta}\right)x^2}$$

This equation represents a parabola and is known as equation of trajectory of a projectile..

Note

The path of a projectile projected at some angle with the horizontal (i.e. ground) is a parabolic path.

Time of Flight

It is defined as the total time for which projectile is in flight i.e. time during the motion of projectile from O to B. It is denoted by T.

Total time of flight consists of two parts such as

(a) Time taken by an object to go from point O to H. It is also known as **time of ascent** (t).

(b) Time taken by an object to go from point H to B. It is also known as **time of descent** (t).

Total time can be expressed as

$$T = t + t = 2t \Rightarrow t = \frac{T}{2}$$

The vertical component of velocity of object becomes zero at the highest point H.

Let us consider vertical upward motion of an object from O to H, we have

$$u_y = u\sin\theta, a_y = -g, t = \frac{T}{2} \text{ and } v_y = 0$$

Since, $v_y = u_y + a_y t \Rightarrow 0 = u\sin\theta - g\frac{T}{2}$

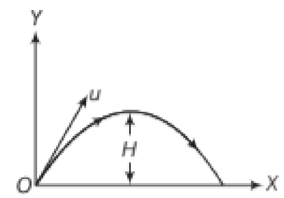
$$\boxed{\text{Total time of flight, } T = \frac{2u\sin\theta}{g}}$$

Note

For a projectile, time of ascent equals time of descent.

Maximum Height of a Projectile

It is defined as the maximum vertical height attained by an object above the point of projection during its flight. It is denoted by H.



Maximum height of motion

Let us consider the vertical upward motion of the object from O to H.

We have,

$$u_y = u\sin\theta, a_y = -g, y_0 = 0, y = H, t = \frac{T}{2} = \frac{u\sin\theta}{g},$$

Using this relation, $y = y_0 + u_y t + \frac{1}{2} a_y t^2$

We have, $H = 0 + u\sin\theta\frac{u\sin\theta}{g} + \frac{1}{2}(-g)\left(\frac{u\sin\theta}{g}\right)^2$

$$= \frac{u^2}{g}\sin^2\theta - \frac{1}{2}\frac{u^2\sin^2\theta}{g}$$

$$\boxed{\text{Maximum height, } H = \frac{u^2\sin^2\theta}{2g}}$$

Horizontal Range of a Projectile

The horizontal range of the projectile is defined as the horizontal distance covered by the projectile during its time of flight. It is denoted by R.

If the object having uniform velocity $u\cos\theta$ (i.e. horizontal component) and the total time of flight T, then the horizontal range covered by the objective.

$$\therefore \quad R = u\cos\theta \times T = u\cos\theta \times 2u\frac{\sin\theta}{g}$$

$$= \frac{u^2}{g} 2\sin\theta\cos\theta$$

$$\boxed{\text{Horizontal range, } R = \frac{u^2 \sin 2\theta}{g}}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

The horizontal range will be maximum, if

$$\sin 2\theta = \text{maximum} = 1$$

$$\sin 2\theta = \sin 90^\circ \text{ or } \theta = 45^\circ$$

$$\therefore \quad \boxed{\text{Maximum horizontal range, } R_m = \frac{u^2}{g}}$$

EXAMPLE |6| Motions of a Soccer Ball

A soccer player kicks a ball at an angle of 30° with an initial speed of 20 m/s. Assuming that the ball travels in a vertical plane. Calculate (i) the time at which the ball reaches the highest point, (ii) the maximum height reached, (iii) the horizontal range of the ball and (iv) the time for which the ball is in the air. $g = 10 \text{ m/s}^2$.

Sol. Given, $\theta = 30^\circ, u = 20 \text{ m/s}, g = 10 \text{ m/s}^2$

(i) $t = ?$ (ii) $H = ?$

(iii) $R = ?$ (iv) $T = ?$

(i) $t = \frac{T}{2} = \frac{u\sin\theta}{g} = \frac{20 \times \sin 30^\circ}{10} = 2 \times \frac{1}{2} = 1 \text{ s}$

(ii) $H = \frac{u^2 \sin^2\theta}{2g} = \frac{(20)^2 \times \sin^2 30^\circ}{2 \times 10} = 5 \text{ m}$

(iii) $R = \frac{u^2 \sin 2\theta}{g} = \frac{(20)^2 \times \sin 2 \times 30^\circ}{10} = 34.64 \text{ m}$

(iv) $T = \frac{2u\sin\theta}{g} = \frac{2 \times 20 \times \sin 30^\circ}{10} = 2 \text{ s}$

EXAMPLE |7| A Pace Bowler

A cricket ball is thrown at a speed of 28 ms^{-1} in a direction 30° above the horizontal. Calculate

(i) the maximum height

(ii) the time taken by the ball to return to the same level and

(iii) the distance from the thrower to the point where the ball returns to the same level. [NCERT]

Sol. (i) The maximum height attained by the ball is

$$H_m = \frac{(v_0 \sin\theta_0)^2}{2g}$$

$$= \frac{(28\sin 30^\circ)^2}{2(9.8)} = \frac{14 \times 14}{2 \times 9.8} = 10.0 \text{ m}$$

(ii) The time taken by the ball to return the same level is

$$T = (2v_0 \sin\theta_0)/g = (2 \times 28 \times \sin 30^\circ)/9.8$$

$$= 28/9.8 = 2.9 \text{ s}$$

(iii) The distance from the thrower to the point where the ball returns to the same level is

$$R = \frac{(v_0^2 \sin 2\theta_0)}{g} = \frac{28 \times 28 \times \sin 60^\circ}{9.8} = 69 \text{ m}$$

Projectile Fired at an Angle with the Vertical

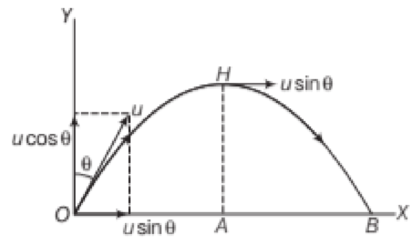
Let a particle be projected vertically with an angle θ with vertical and its muzzle speed (i.e. speed of projection) is u. The projectile has two components of its velocity at all the points during its motion.

The components are along X-axis (horizontal) and along Y-axis (vertical). Clearly, the angle made by the velocity of projectile at point of projection is $(90^\circ - \theta)$ with the horizontal. In this case

(i) Time of flight $= \frac{2u\sin(90^\circ - \theta)}{g} = \frac{2u}{g}\cos\theta$

(ii) Maximum height $= \frac{u^2 \sin^2(90^\circ - \theta)}{2g}$

$$= \frac{u^2 \cos^2\theta}{2g}$$



Angle after projection

(iii) Horizontal range

$$= \frac{u^2}{g} \sin 2(90^\circ - \theta)$$

$$= \frac{u^2}{g} \sin(180^\circ - 2\theta) = \frac{u^2}{g} \sin 2\theta$$

(iv) Path of projectile

$$y = x\tan(90° - \theta) - \frac{1}{2}\frac{gx^2}{u^2\cos^2(90° - \theta)}$$

$$= x\cot\theta - \frac{gx^2}{2u^2\sin^2\theta}$$

(v) Velocity at any time, t

$$= \sqrt{[u\cos(90 - \theta)]^2 + [u\sin(90 - \theta) - gt]^2}$$

$$= \sqrt{u^2 + g^2t^2 - 2ugt\sin(90° - \theta)}$$

$$= \sqrt{u^2 + g^2t^2 - 2u\,gt\cos\theta}$$

This velocity makes an angle β with the horizontal direction, then

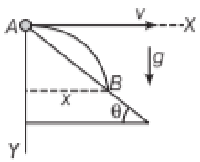
$$\tan\beta = \frac{u\sin(90° - \theta) - gt}{u\cos(90° - \theta)}$$

$$= \frac{u\cos\theta - gt}{u\sin\theta}$$

EXAMPLE |8| Projection of a Particle

A particle is projected horizontally with a speed v from the top of a plane inclined at angle θ with the horizontal. How far from the point of projection will the particle hit the plane?

Sol. To solve this problem, we take X and Y axes as shown in figure.



Consider the motion of particle from A to B.
Motion of particle along X-axis is given by

$$x = ut \qquad \text{...(i)}$$

where, t = time to reach the particle from A to B.
Motion of particle along Y-axis is given by

$$y = \frac{1}{2}gt^2 \qquad \text{...(ii)}$$

Eliminating t from Eqs. (i) and (ii), we get

$$y = \frac{1}{2}g\frac{x^2}{u^2}$$

and $\quad y = x\tan\theta$

Thus, $\frac{gx^2}{2u^2} = x\tan\theta$, giving, $x = 0$ or $\frac{2u^2\tan\theta}{g}$, $x = 0$ at point A and point B correspond to $x = \frac{2u^2\tan\theta}{g}$, then

$$y = x\tan\theta = \frac{2u^2\tan^2\theta}{g}$$

Now, $\quad AB = \sqrt{x^2 + y^2}$

$$= \frac{2u^2}{g}\tan\theta\sqrt{1 + \tan^2\theta} = \frac{2u^2}{g}\tan\theta\sec\theta$$

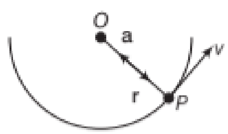
Effect of Air Resistance on Projectile Motion

- As we have seen that in projectile motion, we assume that air resistance has no effect on its motion. Friction, force due to viscosity, air resistance are all dissipative forces.
- A projectile that traverses a parabolic path would slow duration its idealised trajectory in the presence of air resistance. It will not hit the ground with the same speed with which it was projected.
- In the absence of air resistance, it is only the y-component that undergoes a continuous change. However, in the presence of air resistance, both of X and Y-component would get affected.

UNIFORM CIRCULAR MOTION

When an object follows a circular path at a constant speed, the motion of the object is called **uniform circular motion.** The word uniform refers to the speed which is uniform (constant) throughout the motion. Although the speed does not vary, the particle is accelerating because the velocity changes its direction at every point on the circular track.

The figure shows a particle P which moves along a circular track of radius r with a uniform speed u.



A circular motion

Examples

(i) Motion of the tip of the second hand of a clock.

(ii) Motion of a point on the rim of a wheel rotating uniformly.

Terms Related to Circular Motion

Angular Displacement

It is defined as the the angle traced out by the radius vector at the centre of the circular path in the given time. It is denoted by $\Delta\theta$ and expressed in radian. It is a dimensionless quantity. Its a vector quantity, direction is given by Right-hand rule.

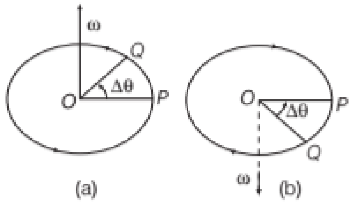
Angular Velocity

It is defined as the time rate of change of its angular position, denoted by ω and is measured in radian per second. Its dimensional formula is $[M^0L^0T^{-1}]$. It is a vector quantity.

If a point object moving along a circular path, with centre (i.e. axis of rotation) at O. Let the object move from P to Q in a small time interval Δt, where $\angle POQ = \Delta\theta$.

Now, angular velocity $\omega = \dfrac{\text{Angle traced}}{\text{Time taken}}$

$$= \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



Time Period

It is defined as the time taken by a particle to complete one revolution along its circular path. It is denoted by T and is measured in second.

Frequency

It is defined as the number of revolutions completed per unit time. It is denoted by f and is measured in Hz.

Angular Acceleration

It is defined as the time rate of change of angular velocity of a particle. It is measured in radian per second square and has dimensions $[M^0L^0T^{-2}]$.

It is denoted by α, where

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Relation between Time Period and Frequency

Let f be the frequency of an object in circular motion, then the object will complete one revolution in $\dfrac{1}{f}$ second which is known as time period (T).

$$\text{Time period, } T = \frac{1}{f}$$

Relation among Angular Velocity, Frequency and Time Period

Suppose a point object illustrating a uniform circular motion with frequency (f) and time period (T). Then, the object completes one revolution, the angle traced at its axis of circular motion is 2π in radian.

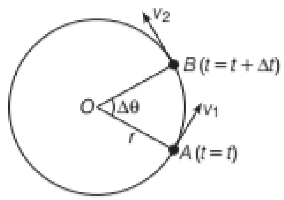
If time $t = T, \theta = 2\pi$ radian.

Thus, the angular velocity ω is given by

$$\text{Angular velocity, } \omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi f$$

Relation between Linear Velocity (v) and Angular Velocity (ω)

Suppose the particle moving on circular track of radius r is showing angular displacement $\Delta\theta$ in Δt time and in this time period, it covers a distance Δs along the circular track, then



Representation of linear velocity and angular velocity

$$\omega = \frac{\Delta\theta}{\Delta t} \quad \text{...(i)}$$

and $$v = \frac{\Delta s}{\Delta t} \quad \text{...(ii)}$$

But $$\Delta s = r\,\Delta\theta \quad \text{...(iii)}$$

(Since, angle = arc/radius)

From above three equations, we get

$$\text{Linear velocity, } v = r\frac{\Delta\theta}{\Delta t} = r\,\omega$$

Centripetal Acceleration

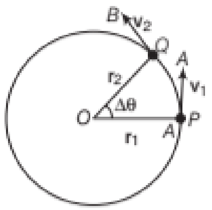
The acceleration associated with uniform circular motion is called a **centripetal acceleration**. Consider a particle of mass (m) moving with a constant speed (v) and uniform angular velocity (ω) on a circular path of radius (r) with centre at O. Suppose at any time t, the particle be at P, where $\mathbf{OP} = r_1$ and at time $t + \Delta t$ the particle be at Q, where $\mathbf{OQ} = r_2$ and $\angle POQ = \Delta\theta$ as shown in the figure. But $|\mathbf{r}_1| = |\mathbf{r}_2| = r$

$$\text{Angular speed of the particle, } \omega = \frac{\Delta\theta}{\Delta t} \quad \text{...(i)}$$

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocity vectors of the particle at locations P and Q respectively. The magnitude and direction of $\mathbf{v}_1$ and $\mathbf{v}_2$ is represented by the tangent **PA** and **QB**.

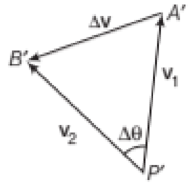
Since the particle is moving with a uniform speed (v), the length of tangents at P and Q are equal

i.e. $$|\mathbf{PA}| = |\mathbf{QB}| = |\mathbf{v}|$$



Centripetal acceleration

These two vectors have been separately shown in the following figures



A triangle made of vectors

From triangle law of vectors, we have

$$\mathbf{P'A'} + \mathbf{A'B'} = \mathbf{P'B'}$$

$$\Rightarrow \quad \mathbf{A'B'} = \mathbf{P'B'} - \mathbf{P'A'}$$

$$= \mathbf{v}_2 - \mathbf{v}_1 = \Delta \mathbf{v}$$

If $\Delta t \to 0$, then A' lies close to B'. Then, $A'B'$ can be taken as an arc $A'B'$ of circle of radius $P'A' = |\mathbf{v}|$

$$\therefore \quad \Delta\theta = \frac{A'B'}{P'A'} = \frac{|\Delta \mathbf{v}|}{|\mathbf{v}|}$$

From Eq. (i), we have

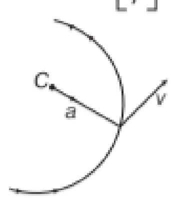
$$\omega \Delta t = \frac{|\Delta \mathbf{v}|}{|\mathbf{v}|}$$

$$\Rightarrow \quad \omega |\mathbf{v}| = \frac{|\Delta \mathbf{v}|}{\Delta t}$$

$$\Rightarrow \quad \frac{|\Delta \mathbf{v}|}{\Delta t} = (\omega r)\,\omega = \omega^2 r \qquad [\because v = \omega r]$$

As $\Delta t \to 0$, then $\frac{|\Delta \mathbf{v}|}{\Delta t}$ represents the magnitude of centripetal acceleration at P which is given by

$$|\mathbf{a}| = \frac{|\Delta \mathbf{v}|}{\Delta t} = \omega^2 r = \left[\frac{v}{r}\right]^2 r = \frac{v^2}{r}$$



Centripetal acceleration, $a = \frac{v^2}{r}$

It is towards the centre of circle.

Radius of Curvature

- Any curved path can be assume to be a part of circular arc. Radius of curvature at a point is defined as the radius of that circular arc which fits at the particular point on the curve as shown in figure.

- In the expression, $a_c = v^2/R$, the term R is known as radius of curvature.

EXAMPLE |9| Trapped Insect

An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s.

(i) What is the angular speed and the linear speed of the motion?

(ii) Is the acceleration vector, a constant vector? What is its magnitude? **[NCERT]**

Sol. The given question is based on uniform circular motion. Here, radius $R = 12$ cm.

The angular speed ω is given as

$$\omega = 2\pi/T = 2\pi \times 7/100 = 0.44 \text{ rad/s}$$

and linear speed v is $v = \omega R = 0.44 \times 12 = 5.3 \text{ cm s}^{-1}$

The direction of velocity v is along the tangent to the circle at every point. The acceleration is directed towards the centre of the circle. Since, this direction changes continuously acceleration, here is not a constant vector.

However, the magnitude of acceleration is constant.

$$a = \omega^2 R = (0.44)^2 \times 12 = 2.3 \text{ cm s}^{-2}$$

EXAMPLE |10| Centripetal Acceleration of a body

A body of mass 10 kg revolves in a circle of diameter 0.4 m making 1000 revolutions per minute. Calculate its linear velocity and centripetal acceleration.

Sol. $m = 10$ kg, $d = 0.4$ m, $r = 0.2$ m,

Revolutions per min, $\nu = 1000/\text{min} = \frac{1000}{60}$ s,

Linear velocity, $v = ?$, centripetal acceleration, $a = ?$

$$\omega = 2\pi\nu = 2\pi \times \frac{1000}{60} = \frac{100\,\pi}{3} \text{ rad/s}$$

$$v = r\omega = 0.2 \times \frac{100\,\pi}{3} = \frac{20\,\pi}{3} \text{ m/s}$$

$$a = r\omega^2 = 0.2 \times \left(\frac{100\,\pi}{3}\right)^2 = \frac{2000\,\pi^2}{9} \text{ m/s}^2$$

TOPIC PRACTICE 2

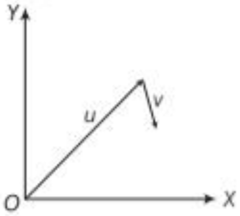
OBJECTIVE Type Questions

1. In a two dimensional motion, instantaneous speed v_0 is a positive constant. Then, which of the following are necessarily true? **[NCERT Exemplar]**

(a) The acceleration of the particle is zero
(b) The acceleration of the particle is bounded
(c) The acceleration of the particle is necessarily in the plane of motion
(d) The particle must be undergoing a uniform circular motion

Sol. (c) As given motion is two dimensional motion and given that instantaneous speed v_0 is positive constant. Acceleration is rate of change of velocity (instantaneous speed) hence it will also be in the plane of motion.

2. Figure shows the orientation of two vectors **u** and **v** in the *xy*-plane. **[NCERT Exemplar]**



If $\mathbf{u} = a\hat{i} + b\hat{j}$ and $\mathbf{v} = p\hat{i} + q\hat{j}$

Which of the following is correct?

(a) a and p are positive while b and q are negative
(b) a, p and b are positive while q is negative
(c) a, q and b are positive while p is negative
(d) a, b, p and q are all positive

Sol. (b) Clearly from the diagram, $\mathbf{u} = a\hat{i} + b\hat{j}$

As **u** is in the first quadrant, hence both components a and b will be positive.

For $\mathbf{v} = p\hat{i} + q\hat{j}$, as it is in positive x-direction and located downward, hence x-component, p will be positive and y-component, q will be negative.

3. Three particles A, B and C projected from the same point with the same initial speeds making angle 30°, 45° and 60°, respectively with the horizontally. Which of the following statements is correct?

(a) A, B and C have unequal ranges
(b) Ranges of A and C are less than that of B
(c) Ranges of A and C are equal and greater than that of B
(d) A, B and C have equal ranges

Sol. (b) When a body is projected at an angle θ with the horizontal with initial velocity u, then the horizontal range R of projectile is $R = \frac{u^2 \sin 2\theta}{g}$.

Clearly, for maximum horizontal range $\sin 2\theta = 1$ or $2\theta = 90°$ or $\theta = 45°$. Hence, in order to achieve maximum range, the body should be projected at 45°.

In this case $R_{max} = \frac{u^2}{g}$

Hence, ranges of A and C are less than that of B.

4. The ceiling of a hall is 30 m high. A ball is thrown with 60 ms^{-1} at an angle θ, so that maximum horizontal distance may be covered. The angle of projection θ is given by

(a) $\sin\theta = \frac{1}{\sqrt{8}}$ (b) $\sin\theta = \frac{1}{\sqrt{6}}$
(c) $\sin\theta = \frac{1}{\sqrt{3}}$ (d) None of these

Sol. (b) Given, $u = 60$ ms^{-1}

$\therefore$ Maximum height, $H = \frac{u^2 \sin^2\theta}{2g}$

$$\Rightarrow \quad 30 = \frac{(60)^2 \sin^2\theta}{2g}$$

$$\Rightarrow \quad \sin^2\theta = \frac{30 \times 2g}{60 \times 60} = \frac{10}{60}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{6}}$$

5. Two cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2, respectively. Their speeds are such that they make complete circles in the same time t. The ratio of their centripetal accelerations is

(a) $m_1 r_1 : m_2 r_2$ (b) $m_1 : m_2$
(c) $r_1 : r_2$ (d) 1:1

Sol. (c) As, centripetal acceleration is given as, $a_c = \frac{v^2}{r}$

For the first body of mass m_1, $a_{c_1} = \frac{v_1^2}{r_1}$

For the second body of mass m_2, $a_{c_2} = \frac{v_2^2}{r_2}$

Also time to complete one revolution by both body is same.

Hence, $\frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2}$

$$\Rightarrow \quad \frac{v_1}{v_2} = \frac{r_1}{r_2} \quad \text{...(i)}$$

i.e., $a_{c_1} : a_{c_2} = \frac{v_1^2}{r_1} \times \frac{r_2}{v_2^2}$ [from Eq. (i)]

$$= \frac{r_1^2}{r_2^2} \times \frac{r_2}{r_1} = \frac{r_1}{r_2} = r_1 : r_2$$

VERY SHORT ANSWER Type Questions

6. Can a body move on a curved path without having acceleration? Why?

Sol. No, a body cannot move on a curved path without acceleration because while moving on a curved path, the velocity of the body changes with time as the direction changes at each point.

7. A particle cannot accelerate if its velocity is constant, why?

Sol. When the particle is moving with a constant velocity, there is no change in velocity with time and hence, its acceleration is zero.

8. The magnitude and direction of the acceleration of a body both are constant. Will the path of the body be necessarily be a straight line?

Sol. No, the acceleration of a body remains constant, the magnitude and direction of the velocity of the body may change.

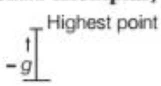
9. Give a few examples of motion in two dimensions.

Sol. A ball dropped from an aircraft flying horizontally, a gun short fired at some angle with the horizontal etc.

10. A football is kicked into the air vertically upwards. What is its (i) acceleration and (ii) velocity at the highest point? [NCERT Exemplar]

Sol. (i) Acceleration at the highest point $= -g$

(ii) Velocity at the highest point = 0.



11. Can there be motion in two dimensions with an acceleration only in one dimension?

Sol. Yes, in a projectile motion, the acceleration acts vertically downwards, while the projectile follows a parabolic path.

12. A stone is thrown vertically upwards and then it returns to the thrower. Is it a projectile?

Sol. No, it is not a projectile, because a projectile should have two component velocities in two mutually perpendicular directions but in this case, the body has velocity only in one direction while going up or coming down.

13. At what point in its trajectory does a projectile have its

(i) minimum speed and (ii) maximum speed?

Sol. (i) Projectile has minimum speed at the highest point of its trajectory.

(ii) Projectile has maximum speed at the point of projection.

14. The direction of the oblique projectile becomes horizontal at the maximum height. What is the cause of it?

Sol. At the maximum height of projectile, the vertical component velocity becomes zero and only horizontal component velocity of projectile is there.

15. Two bodies are projected at an angle θ and $(\pi/2 - \theta)$ to the horizontal with the same speed. Find the ratio of their time of flight.

Sol. The times of flights are

$$T_1 = \frac{2u \sin\theta}{g}$$

and $$T_2 = \frac{2u \sin\left(\frac{\pi}{2} - \theta\right)}{g} = \frac{2u \cos\theta}{g}$$

$\therefore$ $$\frac{T_1}{T_2} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

16. Is the rocket in flight is an illustration of projectile?

Sol. No, because it is propelled by combustion of fuel and does not move under the effect of gravity alone.

17. Why does a tennis ball bounce higher on hills than in plains?

Sol. Maximum height attained by a projectile $\propto 1/g$. As the value of g is less on hills than on plains, so a tennis ball bounces higher on hills than on plains.

18. Galileo, in his book 'Two new sciences', stated that for elevations which exceed or fall short of 45° by equal amounts, the ranges are equal. Prove this statement. [NCERT]

Sol. For a projectile launched with velocity v_0 at an angle θ_0, the range is given by $R = \frac{v_0^2 \sin 2\theta_0}{g}$. Now, for angles, $(45° + \alpha)$ and $(45° - \alpha)$, $2\theta_0$ is $(90° + 2\alpha)$ and $(90° - 2\alpha)$, respectively. The values of $\sin(90° + 2\alpha)$ and $\sin(90° - 2\alpha)$ are the same, equal to that of $\cos 2\alpha$. Therefore, ranges are equal for elevations which exceed or fall short of 45° by equal amount α.

19. A stone tied at the end of string is whirled in a circle. If the string breaks, the stone flies away tangentially. Why?

Sol. When a stone is going around a circular path, the instantaneous velocity of stone is acting as tangent to the circle. When the string breaks, the centripetal force stops to act. Due to inertia, the stone continues to move along the tangent to circular path. So, the stone flies off tangentially to the circular path.

20. **A body is moving on a circular path with a constant speed. What is the nature of its acceleration?**

Sol. The nature of its acceleration is centripetal, which is perpendicular to motion at every point and acts along the radius and directed towards the centre of the curved circular path.

21. **What will be the net effect on maximum height of a projectile when its angle of projection is changed from 30° to 60°, keeping the same initial velocity of projection?**

Sol. As, $H \propto \sin^2\theta$

$$\Rightarrow \quad \frac{H_1}{H_2} = \frac{(\sin 30^\circ)^2}{(\sin 60^\circ)^2} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$$

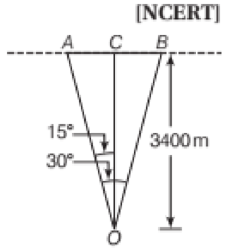
or $H_2 = 3H_1$

The maximum effect of a projectile is three times the initial vertical height.

SHORT ANSWER Type Questions

22. **An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10 s apart is 30°, what is the speed of the aircraft?** **[NCERT]**

Sol. In figure, O is the observation point at the ground, A and B are the positions of aircraft for which $\angle AOB = 30°$. Draw a perpendicular OC on AB. Here $OC = 3400$ m and $\angle AOC = \angle COB = 15°$. Time taken by aircraft from A to B is 10 s. **[1/2]**



In ΔAOC, $AC = OC \tan 15°$

$$= 3400 \times 0.2679$$

$$= 910.86 \text{ m}$$

$$AB = AC + CB = AC + AC = 2AC$$

$$= 2 \times 910.86 \text{ m}$$

Speed of the aircraft

$$v = \frac{\text{distance } AB}{\text{time}} = \frac{2 \times 910.86}{10}$$

$$= 182.17 \text{ ms}^{-1} = 182.2 \text{ ms}^{-1}$$

23. **A passenger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circutious path 23 km long and reaches the hotel in 28 min. What is (i) the average speed of the taxi and (ii) the magnitude of average velocity? Are the two equal?** **[NCERT]**

Sol. Given, shortest distance between the station and the hotel = 10 km

$\therefore$ Displacement of the taxi = 10 km

Distance travelled by the taxi = 23 km

Time taken by the taxi = 28 min = $\frac{28}{60} = \frac{7}{15}$ h

(i) Average speed of the taxi

$$= \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

$$= \frac{23}{(7/15)} = \frac{345}{7} \text{ km/h} = 49.3 \text{ km/h}$$

(ii) Magnitude of average velocity

$$= \frac{\text{Magnitude of the total displacement}}{\text{Total time taken}}$$

$$= \frac{10}{(7/15)} = \frac{150}{7} \text{ km/h} = 21.43 \text{ km/h}$$

No, the average speed of the taxi is not equal to the magnitude of the average velocity of the taxi.

24. **A bullet fired at an angle of 30° with the horizontal hits the ground 3 km away. By adjusting its angle of projection, can one hope to hit a target 5 km away? Assume the muzzle speed to be fixed, and neglect air resistance.** **[NCERT]**

Sol. Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g}$$

or $$3 = \frac{u^2 \sin 60^\circ}{g} = \frac{u^2}{g}\sqrt{3}/2$$

or $$\frac{u^2}{g} = 2\sqrt{3}$$

Since, the muzzle velocity is fixed

Therefore, maximum horizontal range,

$$R_{max} = \frac{u^2}{g} = 2\sqrt{3} = 3.464 \text{km}$$

So, the bullet cannot hit the target.

25. **Find the angle of projection at which horizontal range and maximum height are equal.**

Sol. Horizontal range = Maximum height (given)

$$\therefore \quad \frac{u^2}{g}\sin 2\theta = \frac{u^2 \sin^2\theta}{2g}$$

$$\Rightarrow \quad 2\sin\theta\cos\theta = \frac{\sin^2\theta}{2} \quad [\because \sin 2\theta = 2\cos\theta\sin\theta]$$

$$\tan\theta = 4$$

$$\Rightarrow \quad \theta = 75°58'$$

26. **A football is kicked 20 m/s at a projection angle of 45°. A receiver on the goal line 25 m away in the direction of the kick runs the same instant to meet the ball. What must be his speed, if he has to catch the ball before it hits the ground?**

Sol. Given, $u = 20$ m/s, $\theta = 45°$, $d = 25$ m

Horizontal range is given by

$$R = \frac{u^2}{g}\sin 2\theta = \frac{(20)^2}{9.8}\sin 2(45°)$$

$$= \frac{400}{9.8} \times 1 = 40.82 \text{ m}$$

Time of flight, $T = \frac{2u\sin\theta}{g} = \frac{2\times 20}{9.8}\sin 45°$

$$= 2.886 \text{ s}$$

The goal man is 25 m away in the direction of the ball, so to catch the ball, he is to cover a distance

$= 40.82 - 25 = 15.82$ m in time 2.886 s.

∴ Velocity of the goal man to catch the ball,

$$v = \frac{15.82}{2.886} = 5.48 \text{ m/s}$$

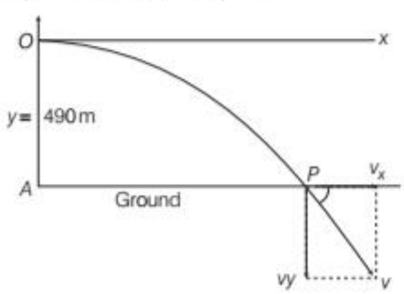
27. **How does the knowledge of projectile help, a player in the baseball game?**

Sol. In the baseball game, a player has to throw a ball so that it goes a certain distance in the minimum time. The time would depend on velocity of ball and angle of throw with the horizontal. Thus, while playing a baseball game, the speed and angle of projection have to be adjusted suitably so that the ball covers the desired distance in minimum time. So, a player has to see the distance and air resistance while playing with a baseball game.

28. **A biker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 m/s. Neglecting air resistance, find the time taken by the stone to reach the ground and the speed with which it hits the ground. Consider $g = 9.8\text{ m/s}^2$.** **[NCERT]**

Sol. Given, $h = 490$ m, $u_x = 15$ m/s,

$a_y = 9.8$ m/s, $a_x = 0$, $u_y = 0$



Time taken by the stone is

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 490}{9.8}} = 10 \text{ s}$$

$$v_x = u_x + a_x t = 15 + 0\times 10 = 15 \text{ m/s}$$

$$v_y = u_y + a_y t = 0 + 9.8\times 10 = 98 \text{ m/s}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 98^2} = 99.14 \text{ m/s}$$

29. **A cricketer can throw a ball to a maximum horizontal distance of 100 m. With the same speed, how high above the ground can the cricketer throw the same ball?** **[NCERT]**

Horizontal range is maximum when angle of projection is 45°.

Sol. Let u be the velocity of projection of the ball. The ball will cover maximum horizontal distance when angle of projection with horizontal, $\theta = 45°$. Then, $R_{max} = u^2/g$.

Here, $u^2/g = 100$ m

In order to study the motion of the ball along vertical direction, consider a point on the surface of Earth as the origin and vertical upward direction as the positive direction of Y-axis. Taking motion of the ball along vertical upward direction, we have

$u_y = u$, $a_y = -g$, $v_y = 0$, $t = ?$, $y_0 = 0$, $y = ?$

As, $v_y = u_y + a_y t$

∴ $0 = u + (-g)t \Rightarrow t = u/g$

Also, $y = y_0 + u_y t + \frac{1}{2}a_y t^2$

∴ $y = 0 + u(u/g) + \frac{1}{2}(-g)u^2/g^2$

$$= \frac{u^2}{g} - \frac{1}{2}\frac{u^2}{g} = \frac{1}{2}\frac{u^2}{g} = \frac{100}{2} = 50 \text{ m}$$

$$\left[\because \frac{u^2}{g} = 100\right]$$

30. **A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, what is the magnitude and direction of acceleration of the stone?** **[NCERT]**

In uniform circular motion, a centripetal acceleration, $a_c = \frac{v^2}{r} = r\omega^2$ acts on the body whose direction is always towards the centre of the path.

Sol. Here, $r = 80$ cm $= 0.8$ m, $f = 14/25\text{ s}^{-1}$

∴ $\omega = 2\pi f = 2\times\frac{22}{7}\times\frac{14}{25} = \frac{88}{25}$ rad/s

The centripetal acceleration

$$a = \omega^2 r = \left(\frac{88}{25}\right)^2 \times 0.80$$

$$= 9.90 \text{ ms}^{-2}$$

The direction of centripetal acceleration is along the string directed towards the centre of circular path.

31. **An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 kmh^{-1}. Compare its centripetal acceleration with the acceleration due to gravity.** **[NCERT]**

Sol. Here, $r = 1\text{ km} = 1000\text{ m}$,

$v = 900\text{ kmh}^{-1} = 900 \times (1000\text{ m})/(60 \times 60\text{ s}) = 250\text{ ms}^{-1}$

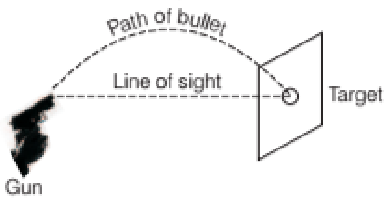
Centripetal acceleration, $a = \frac{v^2}{r} = \frac{(250)^2}{1000}$

Now, $\frac{a}{g} = \frac{(250)^2}{1000} \times \frac{1}{9.8} = 6.38$

32. A skilled gun man always keeps his gun slightly tilted above the line of sight while shooting. Why?

Sol. When a bullet is fired from a gun with its barrel directed towards the target, it starts falling downwards on account of acceleration due to gravity.

Due to which the bullet hits below the target. Just to avoid it, the barrel of the gun is lined up little above the target, so that the bullet after travelling in parabolic path hits the distant target.



33. Prove that the horizontal range is same when angle of projection is

(i) greater than 45° by certain value and

(ii) less than 45° by the same value.

Sol. The horizontal range of the projectile is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

(i) If angle of projection $\theta = 45^\circ + \alpha$ and $R = R_1$

then, $R_1 = \frac{u^2 \sin 2(45^\circ + \alpha)}{g} = \frac{u^2}{g} \cos 2\alpha$ ...(i)

(ii) If angle of projection

$\theta = 45^\circ - \alpha$ and $R = R_2$

$\therefore$ $R_2 = \frac{u^2 \sin 2(45^\circ - \alpha)}{g} = \frac{u^2}{g} \cos 2\alpha$ ...(ii)

Comparing Eqs. (i) and (ii), we have

$R_1 = R_2$

Hence proved.

LONG ANSWER Type I Questions

34. The position of a particle is given by $\mathbf{r} = 3.0t\,\hat{\mathbf{i}} - 2.0\,t^2\hat{\mathbf{j}} + 4.0\,\hat{\mathbf{k}}$ where, t is in seconds and the coefficients have the proper units for $\mathbf{r}$ to be in metres.

(i) Find the $\mathbf{v}$ and $\mathbf{a}$ of the particle.

(ii) What is the magnitude and direction of velocity of the particle at $t = 2$ s? **[NCERT]**

Sol. (i) Velocity, $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(3.0\,t\,\hat{\mathbf{i}} - 2.0\,t^2\hat{\mathbf{j}} + 4.0\,\hat{\mathbf{k}})$

$= [3.0\,\hat{\mathbf{i}} - 4.0\,t\,\hat{\mathbf{j}}]\text{ ms}^{-1}$

Acceleration, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(3.0\,\hat{\mathbf{i}} - 4.0\,t\,\hat{\mathbf{j}})$

$= 0 - 4.0\,\hat{\mathbf{j}}$

$= -4.0\,\hat{\mathbf{j}}\text{ ms}^{-2}$

(ii) At time, $t = 2$ s, $\mathbf{v} = 3.0\,\hat{\mathbf{i}} - 4.0 \times 2\,\hat{\mathbf{j}} = 3.0\,\hat{\mathbf{i}} - 8.0\,\hat{\mathbf{j}}$

$\therefore$ $v = \sqrt{(3.0)^2 + (-8)^2} = \sqrt{73} = 8.54\text{ ms}^{-1}$

If θ is the angle which $\mathbf{v}$ makes with X-axis, then

$\tan\theta = \frac{v_y}{v_x} = \frac{-8}{3} = -2.667 = -\tan 69.5^\circ$

$\therefore$ $\theta = 69.5^\circ$ below the X-axis.

35. The position vector of a particle is

$(\mathbf{r} = 2.0\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + 3.0\hat{\mathbf{k}})$

where, t is in seconds and the coefficients have the proper units for r to be in metres. What will be the value of $\mathbf{v}(t)$ and $\mathbf{a}(t)$ for the particle and the magnitude and direction of $v(t)$ at $t = 2.0$ s? **[NCERT]**

Sol. The position vector, $\mathbf{r} = 2.0\hat{\mathbf{i}}t + t^2\,\hat{\mathbf{j}} + 3.0\,\hat{\mathbf{k}}$

$\therefore$ $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(2.0\hat{\mathbf{i}}\,t + t^2\,\hat{\mathbf{j}} + 3.0\,\hat{\mathbf{k}})$

$\mathbf{v}(t) = 2.0\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}$

$\mathbf{a}(t) = \frac{d\,\mathbf{v}(t)}{dt} = \frac{d}{dt}(2.0\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}) = 2\hat{\mathbf{j}}$

At $t = 2.0$ s, the magnitude of the $\mathbf{v}(t)$ can be given as

$\mathbf{v}(t) = 2.0\,\hat{\mathbf{i}} + 2 \times 2\hat{\mathbf{j}} = 2.0\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$

$\mathbf{v} = \sqrt{4 + 16} = \sqrt{20}$ and direction is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4}{2}\right) = 63^\circ$$

36. A vector has magnitude and direction

(i) Does it have a location in the space?

(ii) Can it vary with time?

(iii) Will two equal vectors $\mathbf{a}$ and $\mathbf{b}$ at different locations in space necessarily have identical physical effects? Give examples in support of your answer. **[NCERT]**

Sol. (i) A vector in general has no definite location in space because a vector remains unaffected whenever it is displaced anywhere in space provided its magnitude and direction do not change. However, a position vector has a definite location in space.

(ii) A vector can vary with time e.g. the velocity vector of an accelerated particle varies with time.

(iii) Two equal vectors at different locations in space do not necessarily have same physical effects. e.g. two equal forces acting at two different points on a body which can cause the rotation of a body about an axis will not produce equal turning effect.

37. Read each statement below carefully and state with reasons, if it is true or false.

(i) The magnitude of a vector is always a scalar.

(ii) Each component of a vector is always a scalar.

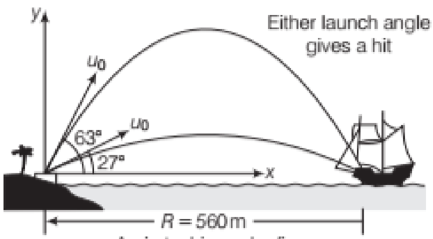
(iii) The average speed of a particle (defined as total path length divided by the time taken to cover the path) is either greater or equal to the magnitude of average velocity of the particle over the same interval of time. [NCERT]

Sol. (i) **True,** because magnitude is a pure number.

(ii) **False,** each component of a vector is also a vector.

(iii) **True,** because the total path length is either greater than or equal to the magnitude of the displacement vector.

38. Figure shows a pirateship 560 m from a fort defending a harbour entrance. A defence cannon, located at sea level, fires balls at initial speed, $u_0 = 82\text{ m/s}$.



A pirateship under fire

(i) At what angle, θ_0 from the horizontal must a ball be fired to hit the ship?

(ii) What is the maximum range of the cannon balls?

A fired cannon ball is a projectile. We relate angle and the horizontal displacement i.e. range as it moves from cannon to ship.

Sol. (i) A fired cannon ball is a projectile and we want an equation that relates the launch angle θ_0 to the ball horizontal displacement i.e. range as it moves from the cannon to the ship.

$$\therefore \quad \theta_0 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{u_0^2}\right) = \frac{1}{2}\sin^{-1}\left(\frac{9.8 \times 560}{(82)^2}\right)$$

$$= \frac{1}{2}\sin^{-1}(0.816) = 27°$$

If one angle is 27°, then other angle $(90° - \theta_0)$ is $= 90° - 27° = 63°$

(ii) Maximum range at $\theta_0 = 45°$

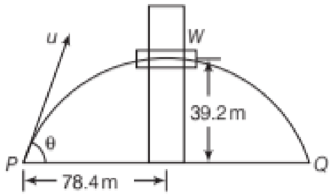
$$\therefore \quad R = \frac{u^2}{g}\sin 2\theta_0 = \frac{(82)^2}{9.8} \times \sin 90°$$

$$= 686\text{ m}$$

39. A boy stands at 78.4 m from a building and throws a ball which just enters a window 39.2 m above the ground. Calculate the velocity of projection of the ball.

Sol. Consider a boy standing at P throw a ball with a velocity u at an angle θ with the horizontal which just enters window W.

As the boy is at 78.4 m from the building and the ball just enters the window 39.2 m above the ground.



$$\therefore \text{ Maximum height, } H = \frac{u^2 \sin^2\theta}{2g}$$

$$\Rightarrow \quad 39.2 = \frac{u^2\sin^2\theta}{2g} \quad \text{...(i)}$$

$$\text{and horizontal range, } R = \frac{u^2 \sin 2\theta}{g}$$

$$\Rightarrow \quad 2 \times 78.4 = \frac{u^2 \sin 2\theta}{g} \quad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{u^2\sin^2\theta}{2g} \times \frac{g}{u^2\, 2\sin\theta\cos\theta} = \frac{39.2}{2 \times 78.4}$$

$$\Rightarrow \quad \frac{1}{4}\tan\theta = \frac{1}{4}$$

$$\Rightarrow \quad \theta = 45°$$

Substituting $\theta = 45°$ in Eq. (ii), we get

$$\frac{u^2 \sin 90°}{9.8} = 2 \times 78.4$$

$$\Rightarrow \quad u = \sqrt{2 \times 78.4 \times 9.8} = 39.2\text{ m/s}$$

40. An aeroplane is flying in a horizontal direction with a velocity of 600 km/h and at a height of 1960 m. When it is vertically above the point A on the ground, a body is dropped from it. The body strikes the ground at point B. Calculate the distance AB.

Find the time taken by the body to fall at the given height

$$y = h = u_{0y}t + \frac{1}{2}gt^2$$

But initial vertical velocity is zero. Then, find the horizontal distance travelled by the body x.

Sol. Velocity of the aeroplane in the horizontal direction is

$$u_{0y} = 600 \text{ km/h} = 600 \times \frac{5}{18} = \frac{500}{3} \text{ m/s}$$

Velocity remains constant throughout the flight of the body.

$$u_{0y} = 0 \text{ and } y = h = 1960 \text{ m}$$

Let t = time taken by the body to reach the ground

Now, $\quad y = u_{0y}t + \frac{1}{2}gt^2$

Here, $\quad y = h = 1960 \text{ m}, u_{0y} = 0$

$\therefore \quad 1960 = \frac{1}{2} \times 9.8 \times t^2$

$\Rightarrow \quad t = \sqrt{\frac{1960}{4.9}} = \sqrt{400} = 20 \text{ s}$

Distance travelled by the body in the horizontal direction,

$$AB = x = v_{ax}\, t = \frac{500}{3} \times 20$$

$$= \frac{10000}{3} = 3333 \text{ m} = 3.33 \text{ km}$$

41. A clever strategy in a snowball fight is to throw two snowballs at your opponent in quick succession. The first one with a high trajectory and the second one with a lower trajectory and shorter time of flight, so that they both reach the target at the same instant. Suppose your opponent is 20.0 m away.

You throw both snowballs with the same initial speed v_0 but θ_0 is 60.0° for the first snowball and 30.0° for the second. If they are both to reach their target at the same instant, how much time must elapse between the release of the two snowballs?

Sol. We need to find the time of flight for each snowball. The time t_R is determined by v_{y0}, the vertical component of initial velocity, then

$$t_R = \frac{2v_{y0}}{g} = \frac{2v_0 \sin\theta_0}{g}$$

To find t_R we need to know, in addition to the initial angle θ_0 (as given), the initial speed v_0, which is not given. We can find v_0 by applying the range equation

$$R = \frac{v_0^2 \sin 2\theta_0}{g}$$

Solving for v_0, we obtain $v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}}$

We obtain the same value for v_0 whether we use,

$$\theta_0 = 30.0° \text{ or } \theta_0 = 60.0°$$

Since, $\sin 2(30.0°) = \sin 2(60.0°)$

$$v_0 = \sqrt{\frac{(20.0 \text{ m})(9.80 \text{ m}/\text{s}^2)}{\sin 60.0°}} = 15.0 \text{ m}/\text{s}$$

Now, we can find t_R for each snowball

$$t_g = \frac{2v_{y0}}{g} \Rightarrow t_g = \frac{2v_0 \sin\theta_0}{g}$$

For the first snowball,

$$t_R = \frac{2\,(15.0 \text{ m}/\text{s} (\sin 60.0°)}{9.80 \text{ m}/\text{s}^2} = 2.65 \text{ s}$$

For the second snow ball,

$$t'_R = \frac{2(15.0 \text{ m}/\text{s})(\sin 30.0°)}{9.80 \text{ m}/\text{s}^2} = 1.53 \text{ s}$$

Thus, you should wait a time Δt before making your second throw, where Δt is the difference in the times of flight, $\quad \Delta t = t_R - t'_R = 2.65 \text{ s} - 1.53 \text{ s} = 1.12 \text{ s}$

42. The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 m/s can go without hitting the ceiling of the hall? **[NCERT]**

Maximum height attained by a projectile is given by $H = \frac{u^2 \sin^2\theta}{2g}$

and horizontal range is given by $R = \frac{u^2 \sin 2\theta}{g}$

Sol. Given, initial velocity $(u) = 40$ m/s

Height of the hall $(H) = 25$ m

Let the angle of projection of the ball be θ, when maximum height attained by it be 25 m.

Maximum height attained by the ball

$$H = \frac{u^2 \sin^2\theta}{2g} \Rightarrow 25 = \frac{(40)^2 \sin^2\theta}{2 \times 9.8}$$

or $\quad \sin^2\theta = \frac{25 \times 2 \times 9.8}{1600} = 0.3063$

or $\quad \sin\theta = 0.5534 = \sin 33.6°$

or $\quad \theta = 33.6°$

$\therefore$ Horizontal range $(R) = \frac{u^2 \sin 2\theta}{g}$

$$= \frac{(40)^2 \sin 2 \times 33.6°}{9.8} = \frac{1600 \times \sin 67.2°}{9.8}$$

$$= \frac{1600 \times 0.9219}{9.8} = 150.5 \text{ m}$$

43. Read each statement below carefully and state, with reasons, if it is true or false.

(i) The net acceleration of a particle in circular motion is always along the radius of the circle towards the centre.

(ii) The velocity vector of a particle at a point is always along the tangent to the path of the particle at that point.

(iii) The acceleration vector of a particle in uniform circular motion averaged over one cycle is a null vector. **[NCERT]**

Sol. (i) False, because the net acceleration of a particle is towards the centre only in case of a uniform circular motion.

(ii) True, because while leaving the circular path, the particle moves tangentially to the circular path.

(iii) True, because the direction of acceleration vector in a uniform circular motion is directed towards the centre of circular path. It is constantly changing with time. The resultant of all these vectors will be a zero vector.

44. A cyclist is riding with a speed of $27\,\text{kmh}^{-1}$. As he approaches, a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate of $0.5\ \text{ms}^{-2}$. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn? **[NCERT]**

Sol. Here, $v = 27\ \text{kmh}^{-1} = 27 \times (1000\ \text{m}) \times (60 \times 60\ \text{s})^{-1}$

$= 7.5\ \text{ms}^{-1}$, $r = 80$ m

Centripetal acceleration, $a_c = \dfrac{v^2}{r} = \dfrac{(7.5)^2}{80} = 0.7\ \text{ms}^{-2}$

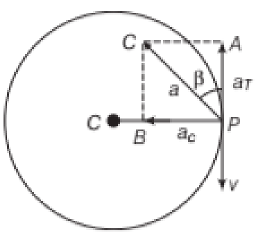
Let the cyclist applies the brakes at the point P of the circular turn, then tangential acceleration a_T will act opposite to velocity.

Acceleration along the tangent, $a_T = 0.5\ \text{ms}^{-2}$

Angle between both the accelerations is 90°

Therefore, the magnitude of resultant acceleration

$$a = \sqrt{a_c^2 + a_T^2} = \sqrt{(0.7)^2 + (0.5)^2} = 0.86\ \text{ms}^{-2}$$



Let the resultant acceleration make an angle β with the tangent i.e. the direction of net acceleration of the cyclist then, $\tan\beta = \dfrac{a_c}{a_T} = \dfrac{0.7}{0.5} = 1.4$ or $\beta = 54°\,28'$

45. A particle starts from the origin at $t = 0$ with a velocity of $10.0\,\hat{\mathbf{j}}$ m/s and moves in the xy-plane with a constant acceleration of $(8.0\,\hat{\mathbf{i}} + 2.0\,\hat{\mathbf{j}})\ \text{ms}^{-2}$.

(i) At what time is the x-coordinate of the particle 16 m? What is the y-coordinate of the particle at that time ?

(ii) What is the speed of the particle at that time? **[NCERT]**

Sol. Here, $\mathbf{u} = 10.0\,\hat{\mathbf{j}}\ \text{ms}^{-1}$ at $t = 0$.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (8.0\,\hat{\mathbf{i}} + 2.0\,\hat{\mathbf{j}})\ \text{ms}^{-2}$$

So, $\quad d\mathbf{v} = (8.0\,\hat{\mathbf{i}} + 2.0\,\hat{\mathbf{j}})\,dt$

Integrating it within the limits of motion i.e. as time changes from 0 to t, velocity changes from u to v, we have

$$\mathbf{v} - \mathbf{u} = (8.0\,\hat{\mathbf{i}} + 2.0\,\hat{\mathbf{j}})\,t$$

$\Rightarrow \quad \mathbf{v} = \mathbf{u} + 8.0t\,\hat{\mathbf{i}} + 2.0t\,\hat{\mathbf{j}}$

As, $\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt} \Rightarrow d\mathbf{r} = \mathbf{v}\,dt$

So, $\quad d\mathbf{r} = (\mathbf{u} + 8.0\,t\,\hat{\mathbf{i}} + 2.0t\,\hat{\mathbf{j}})\,dt$

Integrating it within the conditions of motion i.e. as time changes from 0 to t, displacement is from 0 to r, we have

$$\mathbf{r} = \mathbf{u}\,t + \frac{1}{2} \times 8.0\,t^2\,\hat{\mathbf{i}} + \frac{1}{2} \times 2.0\,t^2\,\hat{\mathbf{j}}$$

or $\quad x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = 10\,\hat{\mathbf{j}}\,t + 4.0\,t^2\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$

$$= 4.0\,t^2\,\hat{\mathbf{i}} + (10\,t + t^2)\,\hat{\mathbf{j}}$$

Here, we have, $x = 4.0\,t^2$ and $y = 10\,t + t^2$

$\therefore \quad t = (x/4)^{\frac{1}{2}}$

(i) At $x = 16$ m, $t = (16/4)^{\frac{1}{2}} = 2$s

$$y = 10 \times 2 + 2^2 = 24\ \text{m}$$

(ii) Velocity of the particle at time t is

$$\mathbf{v} = 10\,\hat{\mathbf{j}} + 8.0\,t\,\hat{\mathbf{i}} + 2.0t\,\hat{\mathbf{j}}$$

When $t = 2$ s, then,

$$\mathbf{v} = 10\,\hat{\mathbf{j}} + 8.0 \times 2\,\hat{\mathbf{i}} + 2.0 \times 2\,\hat{\mathbf{j}} = 16\,\hat{\mathbf{i}} + 14\,\hat{\mathbf{j}}$$

$\therefore$ Speed $= |\mathbf{v}| = \sqrt{16^2 + 14^2} = 21.26\ \text{ms}^{-1}$

46. The maximum height attained by a projectile is increased by 10% by increasing its speed of projection, without changing the angle of projection. What will the percentage increase in the horizontal range?

Sol. As, maximum height, $H = \dfrac{u^2}{2g}\sin^2\theta$

Consider ΔH be the increase in H when u changes by Δu, it can be obtained by differentiating the above equation, we get

$$\Delta H = \frac{2u\,\Delta u \sin^2\theta}{2g} = \frac{2\,\Delta u}{u}H$$

$\Rightarrow \quad \dfrac{\Delta H}{H} = \dfrac{2\,\Delta u}{u}$

Given, % increase in H is 10%, so

$$\frac{\Delta H}{H} = \frac{10}{100} = 0.1 \Rightarrow \frac{2\Delta u}{u} = 0.1$$

As, $\quad R = \dfrac{u^2 \sin 2\theta}{g}$

$\because \quad \Delta R = \frac{2u\,\Delta u}{g}\sin 2\theta \Rightarrow \frac{\Delta R}{R} = \frac{2\,\Delta u}{u} = 0.1$

$\therefore$ % increase in horizontal range $= \frac{\Delta R}{R} \times 100$

$= 0.1 \times 100 = 10\%$

47. The range of a rifle bullet is 1000 m, when θ is the angle of projection. If the bullet is fired with the same angle from a car travelling at 36 km/h towards the target, show that the range will be increased by $142.9\sqrt{\tan\theta}$ m.

When the bullet is fired from the moving car, the horizontal component velocity of the bullet increases with the velocity of car. But the vertical component of the velocity remains uneffected.

Sol. Given, $R = 1000$ m

$\therefore$ Horizontal range of the bullet fired at an angle θ is

$$R = \frac{u^2 \sin 2\theta}{g} \Rightarrow 1000 = \frac{u^2\, 2\sin\theta\cos\theta}{g} \quad \text{...(i)}$$

Bullet is fired from the car moving with 36 km/h i.e.10 m/s, then horizontal component of the velocity of bullet $= u\sin\theta + 10$

Vertical component of the velocity of the bullet $= u\sin\theta$

Then, new range of the bullet is

$$R_1 = \frac{2}{g}(u\sin\theta)(u\cos\theta + 10)$$

$$= \frac{2}{g}u^2\sin\theta\cos\theta + \frac{20}{g}u\sin\theta$$

$$\Rightarrow \quad R_1 = R + \frac{20}{g}u\sin\theta$$

$$\Rightarrow \quad R_1 - R = \frac{20}{g}u\sin\theta \quad \text{...(ii)}$$

From Eq. (i), we have $u = \sqrt{\frac{1000 \times g}{2\sin\theta\cos\theta}}$ ...(iii)

Now, substituting the value of u in Eq. (ii), we get

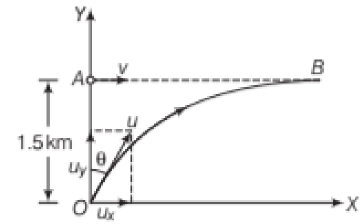
$$R_1 - R = \frac{20}{g}\sqrt{\frac{1000 \times g}{2\sin\theta\cos\theta}}\sin\theta = 20\sqrt{\frac{500 \times \sin\theta}{g\cos\theta}}$$

$$= 20\sqrt{\frac{500}{9.8}\tan\theta} = 142.9\sqrt{\tan\theta}$$

LONG ANSWER Type II Questions

48. A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km h^{-1} passes directly over an anticraft gun. At what angle from the vertical should the gun be fired from the shell with muzzle speed 600 ms^{-1}to hit the plane? At what minimum altitude should the pilot fly the plane to avoid being hit? (Take, g =10 ms^{-2})? [NCERT]

Sol. From the figure, let O be the position of gun and A be the position of plane.



The speed of the plane,

$$v = \frac{720 \times 1000}{60 \times 60} = 200 \text{ ms}^{-1}$$

The speed of the shell, $u = 600$ ms^{-1}

Let the shell hit the plane at B after time t if fired at an angle θ with the vertical from O. Then, the horizontal distance travelled by shell in time t is the same as the distance covered by the plane during the same period.

i.e. $u_x \times t = vt$ or $u\sin\theta\, t = vt$

or $\sin\theta = \frac{v}{u} = \frac{200}{600} = 0.33333 = \sin 19.5°$

or $\theta = 19.5°$ with the vertical

The plane will not be hit by the bullet from the gun if it is flying at a minimum height which is the maximum height (H) attained by bullet after firing from gun.

Here, $H = \frac{u^2\sin^2(90° - \theta)}{2g} = \frac{u^2\cos^2\theta}{2g}$

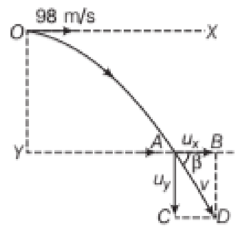
$$= \frac{(600)^2 \times (\cos 19.5°)^2}{2 \times 10}$$

$[\because \sin\theta = 1/3, \cos\theta = \sqrt{8}/3]$

$$= \frac{(600)^2 \times (\sqrt{8}/3)^2}{20} = 16000 \text{ m} = 16 \text{ km}$$

49. A projectile is fired horizontally with a velocity of 98 ms^{-1} from the hill 490 m high. Find (i) time taken to reach the ground (ii) the distance of the target from the hill and (iii) the velocity with which the body strikes the ground.

Sol. Let OX and OY be two perpendicular axes and $YO = 490$ m. A body projected horizontally from O with velocity u (= 98 ms^{-1}) meets the ground at A following a parabolic path shown in figure.



(i) Let T be the time of flight of the projectile i.e. time taken by projectile to go from O to A.

Taking vertical downward motion (i.e. motion along OY axis) of projectile from O to A, we have

$y_0 = 0,\ y = 490$ m, $u_y = 0,\ a_y = 9.8$ m/s^2, $t = T$

As, $\quad y = y_0 + u_y t + \frac{1}{2}a_y t^2$

$\therefore \quad 490 = 0 + 0 \times T + \frac{1}{2} \times 9.8 \times T^2 = 4.9\,T^2$

or $\quad T = \sqrt{\frac{490}{4.9}} = 10$ s

(ii) Taking horizontal motion (i.e. motion along OX axis) of projectile from O to A, we have

$x_0 = 0,\ x = R$ (say), $u_x = 98$ m/s, $t = T = 10$ s, $a_x = 0$

As, $x = x_0 + u_x t + \frac{1}{2} a_x t^2$

$\therefore \quad R = 0 + 98 \times 10 + \frac{1}{2} \times 0 \times 10^2 = 980$ m

(iii) Let v_x, v_y be the horizontal and vertical component velocity of the projectile at A.

Using the relation,

$v_x = u_x + a_x t = 98 + 0 \times 10 = 98$ m/s

Represented by AB

Using the relation,

$v_y = u_y + a_y t = 0 + 9.8 \times 10 = 98$ m/s

Represented by AC.

$\therefore$ Resultant velocity

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{98^2 + 98^2} = 98\sqrt{2} \text{ m/s}$$

If β is the angle which **v** makes with the horizontal direction, then $\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1$ or $\beta = 45°$ with the horizontal.

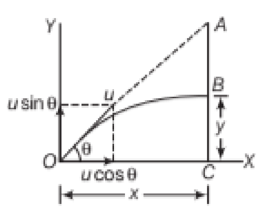
50. A hunter aims his gun and fires a bullet directly at a monkey in a tree. At the instant, the bullet leaves the barrel of the gun, the monkey drops. Will the bullet hit the monkey? Substantiate your answer with proper reasoning.

Sol. Let the monkey stationed at A, be fired with a gun from O with a velocity u at an angle θ with the horizontal direction OX.

Draw AC, perpendicular to OX. Let the bullet cross the vertical line AC at B after time t and coordinates of B (x, y) be w.r.t. origin O as shown in figure.

$$\therefore \quad t = \frac{OC}{u\cos\theta} = \frac{x}{u\cos\theta} \quad \text{...(i)}$$

[where, $OC = x$]



In ΔOAC, $AC = OC \tan\theta = x\tan\theta$ ...(ii)

Clearly, $CB = y =$ the vertical distance travelled by the bullet in time t.

Taking motion of the bullet from O to B along Y-axis, we have $y_0 = 0,\ y = y,\ u_y = u\sin\theta,\ a_y = -g,\ t = t$

As, $\quad y = y_0 + u_y t + \frac{1}{2} a_y t^2$

$\therefore \quad y = 0 + u\sin\theta\, t + \frac{1}{2}(-g)\,t^2$

$= u\sin\theta\, t - \frac{1}{2} g t^2$ ...(iii)

$\therefore AB = AC - BC = x\tan\theta - y$

$= x\tan\theta - \left(u\sin\theta t - \frac{1}{2} g t^2\right)$

$= x\tan\theta - \left(u\sin\theta \times \frac{x}{u\cos\theta} - \frac{1}{2} g t^2\right)$

[from Eq. (i)]

$AB = x\tan\theta - x\tan\theta + \frac{1}{2} g t^2 = \frac{1}{2} g t^2$

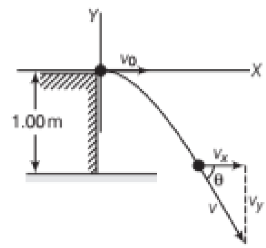
It means the bullet will pass through the point B on vertical line AC at a vertical distance $\frac{1}{2} g t^2$ below point A.

The distance through which the monkey falls vertically in time $t = \frac{1}{2} g t^2 = AB$. It means the bullet and monkey will pass through the point B simultaneously.

Therefore, the bullet will hit the monkey.

51. A marble rolls along a table at a constant speed of 1.00 m/s and then falls off the edge of the table to the floor 1.00 m below. (i) How long does the marble take to reach the floor? (ii) At what horizontal distance from the edge of the table does the marble land? (iii) What is its velocity as it strikes the floor?

Sol. Projectile motion of the marble begins as it leaves the table as shown. Since, the marble is initially moving horizontally, $v_{y_0} = 0$ and $v_{x_0} = 1.00$ m/s. We must consider the origin to be at the edge of the table, so that $x_0 = y_0 = 0$



(i) $\quad t = ?$, if $y = -1.00$ m; $y = \frac{-1}{2} g t^2 \quad [\because v_{x_0} = 0]$

$\Rightarrow \quad t = \sqrt{\frac{-2y}{g}} = \sqrt{\frac{(-2)(-1.00)}{9.8}} = 0.452$ s

(ii) $x = ?$, when $t = 0.452$ s

$\therefore \quad x = v_{x_0} t = 1.00 \times 0.452 \text{ s} = 0.452$ m

(iii) $v = ?, \theta = ?$ at $t = 0.452$ s

The x-component of velocity is constant throughout the motion, $v_x = v_{x_0} = 1.00$ m/s

The y-component of velocity

$$v_y = v_{y_0} - gt = 0 - 9.8 \times .452 = -4.43 \text{ m/s}$$

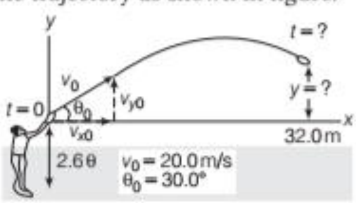
$$\therefore\ v = \sqrt{v_x^2 + v_y^2} = \sqrt{(1.00)^2 + (-4.43)^2} = 4.54 \text{ m/s}$$

$$\theta = \tan^{-1}\left|\frac{v_y}{v_x}\right| = \frac{4.43}{1.00} = 77.3^\circ$$

As the velocity hits the floor, its velocity is 4.54 m/s directed 77.3° below the horizontal.

52. A quarterback, standing on his opponents 35-yard line, throws a football directly down field, releasing the ball at a height of 2.00 m above the ground with an initial velocity of 20.0 m/s, directed 30.0° above the horizontal. (i) How long does it take for the ball to cross the goal line, 32.0 m from the point of release? (ii) The ball is thrown too hard and so passes over the head of the intended receiver at the goal line. What is the ball's height above the ground as it crosses the goal line?

Sol. To better visualise the solution described here, we first sketch the trajectory as shown in figure.



(i) The problem here is to find t when $x = 32.0$ m. We can use $(x = v_{x_0} t)$, if we first find v_{x0}. From figure, we see that $v_{x_0} = v_0 \cos\theta_0 = (20.0 \text{ m/s})(\cos 30.0^\circ)$
$= 17.3$ m/s

Using the relation and solve for t.

$$x = v_{x_0} t$$

$$t = \frac{x}{v_{x0}} = \frac{32.0 \text{ m}}{17.3 \text{ m/s}} = 1.85 \text{ s}$$

(ii) We want to find y when $x = 32.0$ m, or since we have already found the time in part (a), we can state this, find y when $t = 1.85$ s. Using the relation,

$$y = v_{y_0} t - \frac{1}{2} g t^2$$

where $v_{y_0} = v_0 \sin\theta_0 = (20.0 \text{ m/s})(\sin 30.0^\circ)$
$= 10.0$ m/s

Thus, $y = (10.0 \text{ m/s})(1.85 \text{ s}) - \frac{1}{2}(9.80 \text{ m/s}^2)(1.85 \text{ s})^2$

$= 1.73$ m

Since, $y = 0$ is 2.00 m above the ground, this means the ball is 3.73 m above the ground as it crosses the goal line too much high to be caught at that point.

53. (i) Show that for a projectile, the angle between the velocity and the X-axis as function of time is given by

$$\theta_{(t)} = \tan^{-1}\left(\frac{v_{oy} - gt}{v_{ox}}\right)$$

where, the various symbols have their usual meanings.

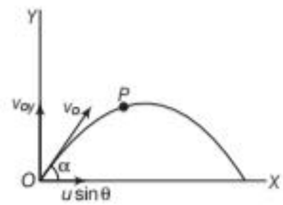
(ii) Show that projection angle θ_0 for a projectile launched from the origin is given by

$$\theta_0 = \tan^{-1}\left(\frac{4h_m}{R}\right)$$

where the symbols have their usual meanings.

[NCERT]

Sol. (i) Let v_{ox} and v_{oy} be the initial component velocities of the projectile at O along OX direction and OY direction respectively, where OX is horizontal and OY is vertical. Let the projectile go from O to P in time t and v_x, v_y be the component velocity of projectile at P along horizontal and vertical directions as shown in figure.



Then, $v_y = v_{oy} - gt$ and $v_x = v_{ox}$

If θ is the angle which the resultant velocity **v** makes with horizontal direction, then

$$\tan\theta = \frac{v_y}{v_x} = \frac{v_{oy} - gt}{v_{ox}}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{v_{oy} - gt}{v_{ox}}\right)$$

(ii) In angular projection,

Maximum vertical height, $h_m = \dfrac{u^2 \sin^2\theta_0}{2g}$

Horizontal range,

$$R = \frac{u^2 \sin^2\theta_0}{g} = \frac{u^2}{g} 2\sin\theta_0 \cos\theta_0$$

So, $\dfrac{h_m}{R} = \dfrac{\tan\theta_0}{4}$ or $\tan\theta_0 = \dfrac{4h_m}{R}$

or $\theta_0 = \tan^{-1}\left(\dfrac{4h_m}{R}\right)$

ASSESS YOUR TOPICAL UNDERSTANDING

OBJECTIVE Type Questions

1. In a two dimensional motion, instantaneous speed v_0 is a positive constant. Then, which of the following are necessarily true? **[NCERT Exemplar]**

(a) The average velocity is not zero at any time
(b) Average acceleration must always vanish
(c) Displacements in equal time intervals are equal
(d) Equal path lengths are traversed in equal intervals

2. A particle starts from origin at $t = 0$ with a velocity $5.0\hat{i}\ ms^{-1}$ and moves in XY-plane under action of force which produces a constant acceleration of $(3.0\hat{i} + 2.0\hat{j})\ ms^{-2}$. What is the y-coordinate of the particle at the instant when its x-coordinate is 84 m?

(a) 36 m (b) 24 m
(c) 39 m (d) 18 m

3. Two projectiles A and B thrown with speeds in the ratio $1 : \sqrt{2}$ acquired the same height. If A is thrown at an angle of 45° with the horizontal, then angle of projection of B will be

(a) 0° (b) 60°
(c) 30° (d) 45°

4. If a person can throw a stone to maximum height of h metre vertically, then the maximum distance through which it can be thrown horizontally by the same person is

(a) $\frac{h}{2}$ (b) h
(c) $2h$ (d) $3h$

5. The displacement of a particle moving on a circular path of radius r when it makes 60° at the centre is

(a) $2r$ (b) r
(c) $\sqrt{2}r$ (d) None of these

6. What is the position vector of a point mass moving on a circular path of radius of 10 m with angular frequency of 2 rads^{-1} after $\pi/8$ s? Initially the point was on Y-axis.

(a) $5\cdot(\hat{i}+\hat{j})$ (b) $5\sqrt{2}(\hat{i}+\hat{j})$
(c) $\hat{i}+\hat{j}$ (d) $\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$

Answer

1. (d)	2. (a)	3. (c)	4. (c)	5. (b)
6. (b)				

VERY SHORT ANSWER Type Questions

7. Show graphically the displacement vector for a motion in two dimensions. Also, write an expression for displacement vector in terms of its rectangular components.

8. A body is projected with speed u at an angle θ to the horizontal to have maximum range. What is the velocity at the highest point? $\left[\textbf{Ans. } \frac{u}{\sqrt{2}}\right]$

9. Is the maximum height attained by projectile is largest when its horizontal range is maximum?

SHORT ANSWER Type Questions

10. A boy can jump on the moon six times as high as on the earth. Why?

11. What will be the effect on maximum height of a projectile when its angle of projection is changed from 30° to 60°, keeping the same initial velocity of projection?

12. A glass marble slides from rest from the top most point of a vertical circle of radius r along a smooth chord. Does the time of descent depend upon the chord chosen?

13. A railway carriage moves over a straight track with acceleration a. A passenger in the carriage drops a stone. What is the acceleration of the stone w.r.t. the carriage and the earth? $[\textbf{Ans. } \sqrt{a^2 + g^2}]$

14. When a rifle is fired at a distant target, the barrel is not lined up exactly on the target. Why?

LONG ANSWER Type I Questions

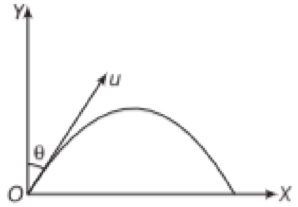
15. The speed of a projectile u reduces by 50% on reaching maximum height. What is the range on the horizontal plane ? $\left[\textbf{Ans. } \frac{u^2}{g} \times \frac{\sqrt{3}}{2}\right]$

16. When a knife is sharpened with the help of a rotating grinding stone, the spark always travel tangentially to it. Why ?

17. An aircraft flying horizontally at a height of 2 km with a speed 200 m/s passes directly over head an anti-aircraft gun. At what angle from the vertical should the gun be fired so that a shell with muzzle speed 600 m/s may hit the plane ? Calculate the safe height of the plane so that the shell may not hit it. (Take, $g = 10 \text{ m/s}^2$) [**Ans.** $\theta = 19.5°$, $H = 16$ km]

18. A projectile is fired at an angle θ with the vertical with velocity u as shown in the figure.

Write the expression for



(i) maximum height (ii) total time of flight (iii) horizontal range

Also, write equation of the path of the projectile.

19. At what angle should a body be projected with a velocity 24 m/s just to pass over the obstacle 16 m high at a horizontal distance of 32 m? (Take, $g = 10 \text{ m/s}^2$) [**Ans.** 67°54′ or 48° 40′]

20. The velocity of a particle, when it is at the greatest height is $\sqrt{2/5}$ times its velocity when it is at half of its greatest height. Determine its angle of projection. [**Ans.** $\theta = 60°$]

LONG ANSWER Type II Questions

21. Derive an expression for horizontal range of a projectile. Also, show that there are two angles of projection for the same horizontal range.

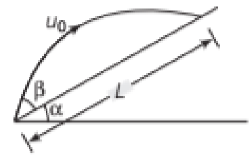
22. A particle is projected in air at an angle β to a surface which itself is inclined at an angle α to the horizontal as shown in figure.

Find (i) time of flight (ii) expression for the range on the plane surface i.e. L and (iii) the value of β at which range will be maximum.

[**Ans.** (i) $T = \dfrac{2u_0 \sin\beta}{g\cos\alpha}$ (ii) $R = \dfrac{2u_0^2 \sin\beta\cos(\alpha+\beta)}{g\cos^2\alpha}$ and (iii) $\beta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$]

23. What is the angular velocity of the hour hand of a clock ?

24. Derive an expression for centripetal acceleration of an object in uniform circular motion in a plane. What will be the direction of the velocity and acceleration at any instant?



SUMMARY

- **Scalar quantities** are quantities with magnitudes only. Examples are distance, speed, mass and temperature.
- **Vector quantities** are quantities with magnitude and direction both. Examples are displacement, velocity and acceleration. They obey special rules of vector algebra.
- A **null** or **zero vector** is a vector with zero magnitude. Since the magnitude is zero, we don't have to specific its direction.
- **Negative vector** Two vectors are said to be negative of each other if their magnitudes are equal but directions are opposite.
- **Collinear vectors** The vectors which either act along the same line or along parallel lines are called collinear vectors.
- **Coplanar vectors** The vectors which act in the same plane are called coplanar vectors.
- **Position vector** A vector which gives position of an object with reference to the origin of a coordinate system is called position vector. It is given by $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$
- **Displacement vector** It is that vector which tells how much and in which direction an object has changed its position in a given time interval.
- **Multiplication of vector by a real number** When a vector **A** is multiplied by a real number λ, we get $\lambda(A) = \lambda A$ and $-\lambda(A) = -\lambda A$.
- **Addition of vectors** Vectors can by added by using Triangle law / Parallelogram law / Polygon law
- **Properties of vector addition**

(i) Vectors representing physical quantities of same nature can only be added.

(ii) Vector addition is **commutative** $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

(iii) Vector addition is **associative** $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

- **Subtraction of vectors** The subtraction of a vector **B** from vector **A** is defined as the addition of vector $-\mathbf{B}$ to **A**.

  Thus, $$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})$$

- **Resolution of a vector** The process of splitting a vector into two or more vectors is known as resolution of the vector.
- A vector **A** can be expressed as $$\mathbf{A}=A_x\hat{\mathbf{i}}+A_y\hat{\mathbf{j}}$$

  where A_x, A_y are its components along x-axis and y-axis. If vector **A** makes an angle θ with the x-axis, then $A_x = A\cos\theta$, $A_y = A\sin\theta$ and $A=|\mathbf{A}|=\sqrt{A_x^2+A_y^2}$, $\tan\theta=\dfrac{A_y}{A_x}$.

- Any vector in three dimensions can be expressed in terms of its rectangular components as
  $$\mathbf{A}=A_x\hat{\mathbf{i}}+A_y\hat{\mathbf{j}}+A_z\hat{\mathbf{k}}$$
  Its magnitude, $A=\sqrt{A_x^2+A_y^2+A_z^2}$

- **Scalar or dot product** The scalar or dot product of two vectors **A** and **B** is defined as the product of the magnitudes of **A** and **B** and cosine of the angle θ between them. Thus $\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos\theta=AB\cos\theta$

  It can be positive, negative or zero depending upon the value of θ.

- **Vector or cross product** For two vectors **A** and **B** inclined at an angle θ, the vector or cross product is defined as
  $$\mathbf{A}\times\mathbf{B}=AB\sin\theta\,\hat{\mathbf{n}}$$
  where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of **A** and **B** and its direction is that in which a right handed screw advances when rotated from **A** to **B**.

- Equations of motion in vector form. For motion with constant acceleration,

  (i) $\mathbf{v}=\mathbf{v}_0+\mathbf{a}t$ (ii) $\mathbf{r}=\mathbf{r}_0+\mathbf{v}_0t+\dfrac{1}{2}\mathbf{a}t^2$ (iii) $v^2-v_0^2=2\mathbf{a}\cdot(\mathbf{r}-\mathbf{r}_0)$

- The motion in a plane with uniform acceleration can be treated as the superposition of two separate simultaneous one-dimensional motions along two perpendicular directions.
- An object that is in flight after being projected is called a projectile.

  The path of a projectile is parabolic and is given by $y=(\tan\theta_0)-\dfrac{gx^2}{2(v_o\cos\theta_o)^2}$

- **Projectile fired at an angle with the horizontal** Suppose a projectile is fired with velocity u at an angle θ with the horizontal. Let it reach the point (x, y) after time t. Then

  (i) Components of initial velocity $u_x=u\cos\theta, u_y=u\sin\theta$

  (ii) Components of acceleration at any instant $a_x=0, a_y=-g$

  (iii) Position after time t, $x=(u\cos\theta)t$, $y=(u\sin\theta)t-\dfrac{1}{2}gt^2$

  (iv) Equation of trajectory: $y=x\tan\theta-\dfrac{g}{2u^2\cos^2\theta}\cdot x^2$

  (v) Maximum height, $H=\dfrac{u^2\sin^2\theta}{2g}$

  (vi) Time of flight, $T=\dfrac{2u\sin\theta}{g}$

  (vii) Horizontal range, $R=\dfrac{u^2\sin 2\theta}{g}$

  (viii) Maximum horizontal range is attained at $\theta=45°$ and its value is $R_{max}=\dfrac{u^2}{g}$

  (ix) Velocity after time t, $v_x=u\cos\theta, v_y=u\sin\theta-gt$

$\therefore \qquad v = \sqrt{v_x^2 + v_y^2}$ and $\tan\beta = \frac{v_y}{v_x}$

(x) The velocity with which the projectile reaches the horizontal plane through the point of projection same as the velocity of projection.

- **Uniform circular motion** When a body moves along a circular path with uniform speed, its motion is said to uniform circular motion.
- **Angular displacement** It is the angle swept out by a radius vector in a given time interval.

$$\theta\,(\text{Rad}) = \frac{\text{Arc}}{\text{Radius}} = \frac{s}{r}$$

- **Angular velocity** The angle swept out by the radius vector per second is called angular velocity.

$$\omega = \frac{\theta}{t} \text{ or } \omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

- **Time period and frequency** Time taken for one complete revolution is called time period (T).

  The number of revolutions completed per second is called frequency.

$$\omega = \frac{2\pi}{T} = 2\,\pi\nu$$

Relationship between v and ω. It is given by $v = r\omega$

- Angular acceleration and its relation with linear acceleration. The rate of change of angular velocity is called angular acceleration. It is given by Linear acceleration $=$ Radius $\times$ Angular acceleration

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}; \quad a = r\alpha$$

- **Centripetal acceleration** A body moving along a circular path is acted upon by an acceleration directed towards the centre along the radius. This acceleration is called centripetal acceleration. It is given by

$$\alpha = \frac{v_2 - v_1}{r} = r\varpi^2$$

CHAPTER PRACTICE

OBJECTIVE Type Questions

1. The angle between $\mathbf{A} = \hat{i} + \hat{j}$ and $\mathbf{B} = \hat{i} - \hat{j}$ is

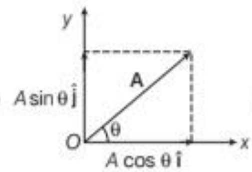
[NCERT Exemplar]

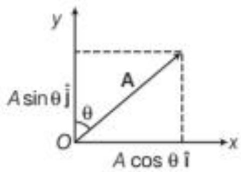
(a) 45° (b) 90° (c) -45° (d) 180°

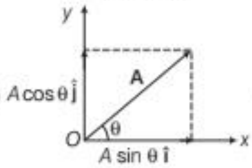
2. The quantities A_x and A_y are called x and y-components of the vector **A**. Note that A_x is itself not a vector, but $A_x\hat{i}$ is a vector, and so is $A_y\hat{j}$. Using simple trigonometry, we can express A_x and A_y in terms of the magnitude of **A** and the angle it makes with the x-axis

$$A_x = A\cos\theta, \quad A_y = A\sin\theta$$

Choose the correct figure on the basis of given description.

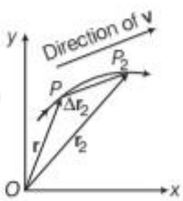
(a)


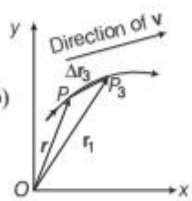
(b)


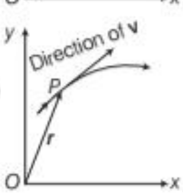
(c)


(d) None of these

3. The direction of instantaneous velocity is shown by

(a)


(b)


(c)


(d) None of these

4. The speed of a projectile at the maximum height is 1/2 its initial speed. Find the ratio of range of projectile to the maximum height attained.

(a) $4\sqrt{3}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{\sqrt{3}}{4}$ (d) 6

5. The horizontal range of a projectile fired at an angle of 15° is 50 m. If it is fired with the same speed at an angle of 45°, its range will be

(a) 60 m (b) 71 m (c) 100 m (d) 141 m

6. Two cars A and B move along a concentric circular path of radius r_A and r_B with velocities v_A and v_B maintaining constant distance, then $\frac{v_A}{v_B}$ is equal to

(a) $\frac{r_B}{r_A}$ (b) $\frac{r_A}{r_B}$ (c) $\frac{r_A^2}{r_B^2}$ (d) $\frac{r_B^2}{r_A^2}$

ASSERTION AND REASON

Direction (Q.Nos. 7-13) In the following questions, two statements are given- one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (a), (b), (c) and (d) as given below

(a) Both Assertion and Reason are true and Reason is the correct explanation of Assertion.
(b) Both Assertion and Reason are true but Reason is not the correct explanation of Assertion.
(c) Assertion is true but Reason is false.
(d) Assertion is false but Reason is true.

7. **Assertion** Force and area are vectors.
Reason Pressure is a vector.

8. **Assertion** Displacement vector is defined with respect to origin.

Reason Position vector is defined with respect to origin.

9. **Assertion** The value of a_x depends on $\frac{dv}{dt}$.

Reason Acceleration means rate of change of velocity.

10. **Assertion** When a body is dropped or thrown horizontally from the same height, it would reach the ground at the same time.

Reason Horizontal velocity has no effect on the vertical direction.

11. **Assertion** The maximum horizontal range of projectile is proportional to square of velocity.

Reason The maximum horizontal range of projectile is equal to maximum height attained by projectile.

12. **Assertion** The range of a projectile is maximum at 45°.
Reason At $\theta = 45°$, the value of $\sin\theta$ is maximum.

13. **Assertion** Speed is constant in uniform circular motion.

Reason Acceleration is constant in uniform circular motion.

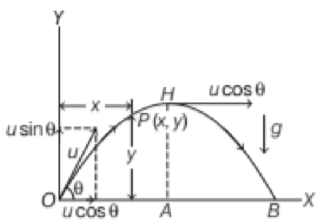
CASE BASED QUESTIONS

Directions (Q. Nos. 14-15) *This questions is case study based question. Attempt any 4 sub-parts from given question.*

14. **Projectile Motion**

Projectile motion is a form of motion in which an object or particle is thrown with some initial velocity near the earth's surface and it moves along a curved path under the action of gravity alone. The path followed by a projectile is called its trajectory, which is shown below.

While resolving velocity (u) into two components, we get (a) $u\cos\theta$ along OX and (b) $u\sin\theta$ along OY.



(i) The example of such type of motion is
(a) motion of car on a banked road
(b) motion of boat in sea
(c) a javelin thrown by an athlete
(d) motion of ball thrown vertically upward

(ii) The acceleration of the object in horizontal direction is
(a) constant (b) decreasing
(c) increasing (d) zero

(iii) The vertical component of velocity at point H is
(a) maximum
(b) zero
(c) double to that at O
(d) equal to horizontal component

(iv) A cricket ball is thrown at a speed of 28 m/s in a direction 30° with the horizontal.
The time taken by the ball to return to the same level will be
(a) 2.0 s (b) 3.0 s
(c) 4.0 s (d) 2.9 s

(v) In above case, the distance from the thrower to the point where the ball returns to the same level will be
(a) 39 m (b) 69 m (c) 68 m (d) 72 m

15. **Uniform Circular Motion**

When an object follows a circular path at a constant speed, the motion of the object is called uniform

circular motion. The word uniform refers to the speed which is uniform (constant) throughout the motion. Although, the speed does not vary, the particle is accelerating because the velocity changes its direction at every point on the circular track.

The figure shows a particle P which moves along a circular track of radius r with a uniform speed v.

(i) A circular motion
(a) is one-dimensional motion
(b) is two-dimensional motion
(c) it is represented by combination of two variable vectors
(d) Both (b) and (c)

(ii) The displacement of a particle moving on a circular path when it makes 60° at the centre is
(a) $2r$ (b) r
(c) $\sqrt{2}r$ (d) None of these

(iii) Two cars A and B move along a concentric circular path of radius r_A and r_B with velocities v_A and v_B maintaining constant distance, then $\frac{v_A}{v_B}$ is equal to

(a) $\frac{r_B}{r_A}$ (b) $\frac{r_A}{r_B}$ (c) $\frac{r_A^2}{r_B^2}$ (d) $\frac{r_B^2}{r_A^2}$

(iv) A particle is moving with a constant speed v in a circle. What is the magnitude of average velocity after half rotation?
(a) $2v$ (b) $\frac{2v}{\pi}$ (c) $\frac{v}{2}$ (d) $\frac{v}{2\pi}$

(v) What is the centripetal acceleration of a point mass which is moving on a circular path of radius 5 m with speed 23 ms^{-1}?
   (a) 106 ms^{-2}
   (b) 90 ms^{-2}
   (c) 60 ms^{-2}
   (d) None of the above

Answer

1. (b)	2. (a)	3. (c)	4. (b)	5. (c)
6. (b)	7. (c)	8. (d)	9. (d)	10. (a)
11. (c)	12. (c)	13. (c)		

14.	(i) (c)	(ii) (a)	(iii) (b)	(iv) (d)	(v) (b)
15.	(i) (d)	(ii) (b)	(iii) (b)	(iv) (b)	(v) (a)

VERY SHORT ANSWER Type Questions

16. When the component of a vector **A** along the direction of **B** is zero, what can you conclude about the two vectors?

17. Displacement vector is fundamentally a position vector. Comment on this statement.

18. Is it necessary to mention the direction of vector having zero magnitude?

19. Does the nature of a vector change when it is multiplied by a scalar? Explain with example.

20. Draw the conclusion about B if A − B = A + B.

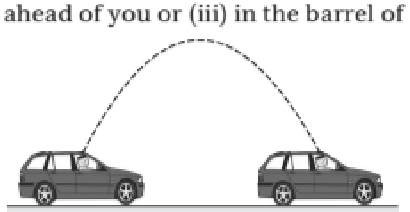
21. For what angle between *P* and *Q*, the value of P + Q is maximum?

22. Can the walk of a man be an example of resolution of vectors? If yes, how?

23. Can there be two vectors, where the resultant is equal to either of them?

SHORT ANSWER Type Questions

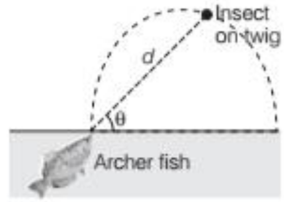
24. Suppose you are driving in a convertible car with the top removed. The car is moving to the right at a constant velocity. As the figure illustrates, you point a toy rifle straight upward and trigger it. In the absence of air resistance, where would the bullet land (i) behind you, (ii) ahead of you or (iii) in the barrel of the rifle?



25. A rabbit runs across a parking lot on which a set of coordinates axes has been drawn. The coordinates (in metres) of the rabbit's position as functions of time *t* (in seconds) are given by

$x = -0.31t^2 + 7.2t + 28$ and $y = 0.22t^2 - 9.1t + 30$

At $t = 15$ s, what is the rabbit's position vector **r** in unit vector notation and in magnitude angle notation?



(i) Draw a graph for rabbit's path for $t = 0$ to 25 s.
(ii) Find the rabbit's velocity v at time $t = 15$ s.
(iii) Find the rabbit's acceleration ***a*** at time $t = 15$ s.

[**Ans.** (ii) $v = 3.3$ m/s, $\theta = -130°$
(iii) $a = 0.76$ m/s^2, $\theta = -145°$]

26. When a large star becomes a supernova, its core may be compressed so tightly that it becomes a neutron star, with a radius of about 20 km. If a neutron star rotates once every second, (i) what is the speed of a particle on the star's equator and (ii) what is the magnitude of the particle's centripetal acceleration? (iii) If the neutron star rotates faster, do the answer to (i) and (ii) increase, decrease or remain the same?

[**Ans.** (i) $a = 1.3 \times 10^5$ m/s, (ii) 7.9×10^5 m/s^2 and (iii) increase]

27. A woman rides a carnival ferris wheel at radius 15 m, completing five turns about its horizontal axis every minute. What are (i) the period of the motion, (ii) the magnitude and (iii) direction of her centripetal acceleration at the highest point (iv) magnitude and (v) direction of her centripetal acceleration at the lowest point?

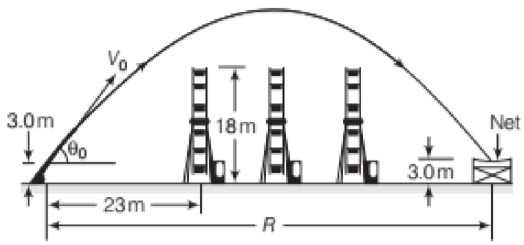
[Ans. (i) 12s, (ii) 4.1 m/s^2, (iii) down and (iv) up]

28. A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above level ground. The string breaks, and the stone flies off horizontally and strikes the ground after travelling a horizontal distance of 10 m.

What is the magnitude of the centripetal acceleration of the stone during the circular motion?

[**Ans.** 160 m/s^2]

29. Around 1939-1940, Emanuel Zacchini took human-cannon ball act to an extreme. After being shot from a cannon, it soared over three Ferris wheels and into a net as shown in figure. Assume that it is launched with a speed of 26.5 m/s and at an angle of 53.0°. (i) Treating it as a particle, calculate its clearance over the first wheel. (ii) If he reached maximum height over the middle wheel, by how much did he clear it? (iii) How far from the cannon should the net's centre have been positioned (neglect air drag)?



[**Ans.** (*a*) 5.3 m, (*b*) 7.9 m and (*c*) 69 m]

30. Upon spotting an insect on a twig overhanging water, an archer fish squirts water drops at the insect to knock it into the water as shown in figure. Although the fish sees the insect along a straight-line path at angle ϕ and distance d a drop must be launched at a different angle θ_0. if its parabolic path is to intersect the insect.

If $\phi = 36.0°$ and $d = 0.900$ m, what launch angle θ_0 is required for the drop to be at the top of the parabolic path when it reaches the insect?

[**Ans.** $\theta = 55.5°$]

31. A dart is thrown horizontally with an initial speed of 10 m/s toward point *P*, the bull's eyes on a dart board. It hits at point *Q* on the rim, vertically below *P*, 0.19 s later. (i) What is the distance *PQ*? (ii) How far away from the dart board is the dart released?

[**Ans.** (i) 18 cm and (ii) 1.9 m]

LONG ANSWER Type I Questions

32. A rifle that shoots bullets at 460 m/s is to aim at a target 45.7 m away. If the centre of the target is level with the rifle, how high above the target must the rifle barrel be pointed so that the bullet hits dead centre? [**Ans.** 4.84 cm]

33. Water from a sprinkler comes out with a constant velocity u in all the directions. What is the maximum area of the grassland that can be watered at any time? [**Ans.** $\pi u^4/g^2$]

34. What is the speed of an aircraft if the pilot remains in contact with the seat, even while looping in vertical plane?

35. Prove that the path of one projectile as seen from another projectile is a straight line.

36. A bullet *P* is fired from a gun when the angle of elevation of the gun is 30°. Another bullet *Q* is fired from the gun when the angle of elevation is 60°. The vertical height attained in the second case is x times the vertical height attained in the first case. What is the value of x ? [**Ans.** 3]

37. Two billiard balls are rolling on a flat table. One has the velocity components $v_x = 1\ ms^{-1}$, $v_y = \sqrt{3}\ ms^{-1}$ and the other has components $v'_x = 2\ ms^{-1}$ and $v'_y = 2\ ms^{-1}$. If both the balls start moving from the same point, what is the angle between their paths? [**Ans.** 15°]

38. If R is the horizontal range for θ inclination and h is the maximum height reached by the projectile, show that the maximum range is given by $\frac{R^2}{8h} + 2h$.

LONG ANSWER Type II Questions

39. A hunter aims his gun and fires a bullet directly at a monkey on a tree. At the instant, the bullet leaves the barrel of the gun, the monkey drops. Will the bullet hit the monkey? Substantiate your answer with proper reasoning.

40. While firing, one has to aim a little above the target and not exactly on the target. Explain.

41. A ball rolls of the top of a stairway with horizontal velocity of 1.8 m/s. The steps are 0.24 m high and 0.2 m wide. Which step will the ball hit first? Take, $g = 9.8\ m/s^2$ [**Ans.** Fourth step]

42. Justify that a uniform circular motion is an accelerated motion.

43. A particle is thrown over a triangle from one end of a horizontal base that grazing the vertex falls on the other end of the base. If α and β be the base angles and θ be the angle of projection, then show that $\tan\theta = \tan\alpha + \tan\beta$

44. A machine gun is mounted on the top of a tower 100 m high. At what angle should the gun be inclined to cover a maximum range of firing on the ground below? The muzzle speed of the bullet is 150 m/s. Take, $g = 10\ m/s^2$

[**Hint** $R = u_x \times t = 150\cos\theta\,(15\sin\theta + \sqrt{225\sin^2\theta + 20})$]

[**Ans.** $\theta = 44°$]